Moments of generalized order statistics

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Abstract

We consider the generalized order statistics based on a given distribution function with no restrictions on the parameters. Under the assumption of finiteness of the moment of a fixed order of the parent distribution function,we present sufficient conditions for existence of the moments of various orders of the generalized order statistics in terms of the model parameters. We provide mean bounds on the expectations of generalized order statistics of nonnegative populations. We also evaluate optimally the deviations of the expectations of the generalized order statistics from the population mean in terms of various scale units.

1 Preliminaries

Generalized order statistics $X_r = X(r, n, \gamma_1, \dots \gamma_n)$, $1 \le r \le n < \infty$, based on a distribution function F, with positive parameters $\gamma_1, \dots, \gamma_n$, were defined and extensively studied in Kamps (1995). They provide a useful general description of ordered statistical data models, including ordinary, noninteger, sequential, progressively type II censored order statistics, standard, kth and Pfeifer's records. Cramer and Kamps (2001) proved that the marginal distribution function of X_r is

$$F_r(F(x)) = 1 - \left(\prod_{i=1}^{r} \gamma_i\right) \int_0^{1-F(x)} \mathbf{G}_{r,r}^{r,0}\left[t \,\middle|\, \begin{matrix} \gamma_1, \ \dots, \ \gamma_r \\ \gamma_1-1,\dots,\gamma_r-1 \end{matrix} \right] dt,$$

where $\mathbf{G}_{r,r}^{r,0}\left[\cdot \,\middle|\, \begin{matrix} \gamma_1, \ \dots, \ \gamma_r \\ \gamma_1-1,\dots,\gamma_r-1 \end{matrix} \right]$ is the Meijer G-function (cf. Mathai, 1993) with particularly chosen parameters. In particular, the standard uniform rth generalized order statistic has the distribution function $F_r(x)$ and density function

$$f_r(x) = \left(\prod_{i=1}^{r} \gamma_i\right) \mathbf{G}_{r,r}^{r,0}\left[1-x \,\middle|\, \begin{matrix} \gamma_1, \ \dots, \ \gamma_r \\ \gamma_1-1,\dots,\gamma_r-1 \end{matrix} \right], \quad x \in [0,1), \tag{1}$$

which is continuously differentiable positive on $(0,1)$, and vanishes elsewhere. Due to (1), the rth uniform generalized order statistic has the distribution identical with that of the product of independent power random variables with distribution functions $G_i(x) = x^{\gamma_i}$, $1 \le i \le r$, subtracted from 1 (cf. Mathai, 1993, pp. 83–84). The generalized order statistic based on arbitrary F can be derived from the uniform one by the quantile transformation F^{-1}. Since we consider only marginal distributions of single generalized order statistics, without loss of generality we assume that $\gamma_1 \geq \ldots \geq \gamma_r > 0$. No other restrictions on the parameters are imposed. For $r = 1$, we have,

$$f_1(x) = \gamma_1(1-x)^{\gamma_1 - 1}, \quad 0 < x < 1,$$

is bounded increasing, constant, and decreasing for $\gamma_1 < 1$, $= 1$, > 1, respectively. For $r \geq 2$, Cramer et al (2002a) proved the following.

Theorem 1 *If $\gamma_r \leq 1$, then the density function is strictly increasing. Otherwise it is strictly unimodal with a mode in $(0,1)$. Moreover,*

$$\begin{aligned} \lim_{t \to 0+} f_r(t) &= 0, \\ \lim_{t \to 1-} f_r(t) &= \begin{cases} +\infty & if \;\; \gamma_r < 1 \;\; or \;\; \gamma_r = \gamma_{r-1} = 1, \\ f_r(1) \in (0, \infty), & if \;\; \gamma_r = 1 < \gamma_{r-1}, \\ 0, & if \;\; \gamma_r > 1. \end{cases} \end{aligned}$$

The finite value of $f_r(1)$ in the middle row has a complicated form dependent on γ_i, $1 \leq i \leq r$, and can be found in Cramer et al (2002a). Theorem 1 enables us to provide sufficient conditions for existence of moments of generalized order statistics and sharp bounds on their expectations expressed in various scale units, presented in Sections 2 and 3, respectively. To this end, the representation

$$\mathrm{E}G(X_r) = \int_0^1 G(F^{-1}(t)) f_r(t) dt$$

is used, together with the above properties of the density function (1).

2 Existence of moments

Theorem 2 *Let X have a distribution function F such that $\mathrm{E}|X|^\beta < \infty$ for some $\beta > 0$, and let X_r denote the rth generalized order statistic based on F.*

(i) For all $0 < \alpha < \beta \min\{\gamma_r, 1\}$, we have

$$\mathrm{E}|X_r|^\alpha < \infty. \tag{2}$$

(ii) If

$$(r = 1 \wedge \gamma_1 \geq 1) \vee [r \geq 2 \wedge (\gamma_r = 1 < \gamma_{r-1} \vee \gamma_r > 1)],$$

then (2) holds with $\alpha = \beta$.

(iii)Relation (2) holds for all $\alpha > \beta$ satisfying

$$(r \geq \alpha/\beta) \wedge [(\gamma_r = \alpha/\beta < \gamma_{r-1}) \vee (\gamma_r > \alpha/\beta)].$$

Assumption (i) is satisfied by record values, ordinary and progressively type II censored order statistics. Assumptions (ii) do not hold for the standard first records. Ordinary order statistics satisfy (iii) if

$$\alpha/\beta \leq r \leq n + 1 - r - \alpha/\beta$$

(cf. Sen, 1959). For rth values of kth order statistics, they are equivalent to $r \geq \alpha/\beta$ with $k > \alpha/\beta$, which is impossible for the first records. Progressively type II censored order statistics satisfy (iii) if either

$$\alpha/\beta \leq r \leq N + 1 - \sum_{i=1}^{r-1} R_i - \alpha/\beta$$

or

$$\alpha \leq \beta \min\{r, N + 1 - r - \sum_{i=1}^{r-1} R_i\}.$$

3 Bounds

We assume that X has finite mean μ and absolute central moment $\sigma_p^p = \mathrm{E}|X-\mu|^p$ for some $p \in [1,\infty)$. If X is bounded, we also define $\sigma_\infty = \operatorname{ess\,sup}|X-\mu|$. Below we present the sharp upper bounds on $\mathrm{E}(X_r-\mu)/\sigma_p$ for $p = 1, 2, \infty$. More complicated formulae for general $1 \le p \le \infty$ can be found in Cramer et at (2002b). Distributions which attain the bounds are precisely described there as well. If $\gamma_1 \ge 1$, then $\mathrm{E}X_1 \le \mu$ for all parent distributions F. Otherwise we consider two cases:

$$(r = 1 \vee \gamma_1 < 1) \wedge (r \ge 2 \vee \gamma_r \le 1), \tag{3}$$

and

$$r \ge 2 \vee \gamma_r > 1. \tag{4}$$

Note that the density function (1) is increasing and strictly unimodal in cases (3) and (4), respectively.

Theorem 3 *(i) If* (3) *holds, then*

$$\sup_F \frac{\mathrm{E}X_r - \mu}{\sigma_p} = \begin{cases} \frac{f_r(1)-f_r(0)}{2}, & if\ p=1, \\ \left[\int_0^1 f_r^2(t)dt - 1\right]^{1/2}, & if\ p=2, \\ 1-2F_r\left(\frac{1}{2}\right), & if\ p=\infty. \end{cases} \tag{5}$$

(ii) If (4) *holds, then there exists a unique* $x \in (0,1)$ *satisfying*

$$(1-x)f_r(x) = 1 - F_r(x) \tag{6}$$

such that

$$\sup_F \frac{\mathrm{E}X_r - \mu}{\sigma_p} = \begin{cases} \frac{f_r(x)}{2}, & if\ p=1, \\ \left[\int_0^x f_r^2(t)dt + (1-x)f_r^2(x) - 1\right]^{1/2}, & if\ p=2, \\ f_r(x)-1, & if\ p=\infty\ and\ x \le 1/2, \\ 1-2F_r\left(\frac{1}{2}\right) - 2(1-x)f_r(x), & if\ p=\infty\ and\ x > 1/2. \end{cases} \tag{7}$$

The first bounds in (5) and (7) are attained in limit by some sequences of three-point distributions. The second ones become equalities for the distributions whose quantile functions are affine transformations of f_r, with the jump of height $1-x$ at the right end-point of the support in case (4). The remaining bounds are attained by two-point distributions.

Finally we present sharp lower and upper mean bounds for the generalized order statistics based on distributions supported on the positive halfaxis. The results come from Cramer et al (2002a).

Theorem 4 *(i) If* (3) *holds, then*

$$1 \le \frac{\mathrm{E}X_r}{\mu} \le f_r(1). \tag{8}$$

(i) If (4) *holds, then*

$$0 \le \frac{\mathrm{E}X_r}{\mu} \le f_r(x) \tag{9}$$

for $x \in (0,1)$ *defined in* (6) .

If either $r \ge 1$ with $\gamma_r < 1$ or $r \ge 2$ with $\gamma_{r-1} = \gamma_r = 1$, then $f_r(1) = +\infty$, and the upper bounds in the first row of (5) and in (8) are infinite. If $r \ge 2$ with $\gamma_{r-1} > \gamma_r = 1$, the bounds are finite. The lower bound in (8) and the upper one in (9) become equalities for degenerate and two-point distributions, respectively. The other ones are attained in limit for distributions with three atoms. The solutions to (6) and respective bounds are determined numerically.

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