On multi-state reliability systems *

Boyan Dimitrov *bdimitro@kettering.edu*
Vladimir Rykov *vrykov@kettering.edu*
Peter Stanchev *pstanche@kettering.edu*
Dept. of Science and Mathematics, Kettering University
1700 West Third Ave., Flint, MI 48504, USA

Abstract

A general model for describing, evaluation and control of most common reliability characteristics of multi-stage systems with gradual failures and repairs of the whole system is proposed. The steady state probabilities and the Laplace transform of the reliability function are calculated.

1 Introduction and Motivation

In complex systems with controllable reliability any complete system failure does not occur suddenly but usually is a result of accumulation of a sequence of many gradual failures. It stimulates consideration of systems with gradual failures of different types, or multi-state reliability systems (for an extensive recent bibliography, please, see Levitin, 2001).

In this paper we propose a general approach to describe, model and evaluate the most common reliability characteristics of systems with various types of gradual failures. Such failures may change the state of the system and the quality of its operation, but do not necessarily lead to complete system failure. Only failures at the last stage of a component (called later "blocks") cause system failure. Two main characteristics are common in the reliability studies: the life-time of the system, and its steady state characteristics under some assumptions about repair process. The ways to evaluate these characteristics depend on the approach to the following two aspects: probabilistic and structural. Probabilistic aspect deals with calculation of the system states probabilities, and uses them in reliability calculations. The structural aspect considers kind of direct evaluation of reliability characteristics for any given structure of a particular system.

In this paper we deal with probabilistic aspect of modeling system reliability and focus on both of its common characteristics. In next section a general model for describing reliability process in systems with gradual failures is proposed. An explicit study for a simple unit model is given in section 3. Its extension to the general model is shown in section 4. Some examples are given at the conference presentation.

2 A General Model

Consider complex hierarchical multi-component system subjected to gradual (internal) failures of different types. Assume that the system is constructed from blocks and branches of several, let say L levels. Each block and the following after branches and blocks forms a hierarchical subsystem of the same type as the main one. The blocks of the last (lowest) level will be referred as units and also may be subjected to gradual failures of its own type. The reliability of each unit is partially controllable. In case of detection of a failure the *whole system* is returned to its initial state (e.g. by replacing with a new one of the same type).

To specify the state space of the system and to define appropriate process describing its behavior we introduce vector index $k = (i_1, i_2, \ldots, i_L)$ which determines each unit of the system as belonging to an appropriate chain of blocks at any level. Denote also by $\mathcal{K}$ the set of these indices (and appropriate units). Then the state space of the system can be represented as $E = \{\mathbf{x} = (x_k : k \in \mathcal{K})\}$, where for any $k \in \mathcal{K}$ the integer x_k represents the state of the k-th unit in sense of its reliability. It can take different values, depending on its type, $x_k \in \{0, 1, \ldots n_k\}$, where the exhausted level of k-th unit is denoted by n_k. Notice, that these numbers have no specific physical sense, but indicate only a possible level of gradual failure of the k-th unit. The value of $x_k = n_k$ means the full failure of k-th unit. Denote also by Γ the set of all boundary states, for which at least one of the components x_k takes value n_k $(k \in \mathcal{K})$, and by F the set of the full system breakdown states.

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Remark. In some cases it may be convenient to consider more complicated units, subjected to different types of gradual failures. In such a case it could be possible to supply the state of a unit with some additional index, say j, to indicate the type of gradual failures of this unit from a set $\mathcal{J}_k$. In this case the state space expands to: $E = \{\mathbf{x} = (x_{kj} : \ j \in \mathcal{J}_k, \ k \in \mathcal{K})\}$. It may change the structural analysis of the system only, and will not change its probabilistic analysis in principal.

To model the system as functioning according to a finite state Markov process we assume that the times of transition from one gradual level to another, as well as the times to repair a failed unit have exponential distributions. The respective parameters may depend on the type of the unit $k \in \mathcal{K}$ and also on the entire system state $\mathbf{x}$. These assumptions allow us to model the reliability of such a system by using the multi-dimensional Markov process

$$\{\mathbf{X}\} = \{X_k(t) : \ k \in \mathcal{K}, \ t \geq 0\},$$

with set of states: E, which should be specified for any particular system.

Additional assumption concerns the structure of transition intensities of such a process. The specifics of the reliability models make it reasonable to suppose that this process can jump only in neighboring states (in case that gradual failure arise, then a transition into a next level will occur), and a transition into the initial state takes place (if the system is repaired). This means that the transition intensities have the following form

$$a(\mathbf{x}, \mathbf{y}) = \begin{cases} \lambda_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \\ \mu_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{0}, \end{cases} \tag{1}$$

with $\lambda_k(\mathbf{x}) = 0$ for $x \in \Gamma$ and any $k \in \mathcal{K}$ and $\mu(\mathbf{0} = 0$. Here and later the following notations are used:

$$\begin{aligned} \mathbf{e}_k & \quad \text{a unit vector with 1 at } k\text{-th position and zeros elsewhere,} \\ \gamma_k(\mathbf{x}) &= \lambda_k(\mathbf{x}) + \mu_k(\mathbf{x}), \quad \text{for } \mathbf{x} \notin \Gamma, \text{ and } \quad \gamma_k(\mathbf{x}) = \mu_k(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma. \end{aligned} \tag{2}$$

We will refer to processes having these properties in possession as to *a Multi-failed Reliability Process* (MRP). The intensities $\lambda_k(\mathbf{x})$ and $\mu_k(\mathbf{x})$ will be called *failure* and *repair intensities* correspondingly. Different constrains to the states space E, and/or the failure set F of this process and various assumption about dependence of transition intensities on the state give the possibility to model a number of particular cases. We begin with the simple unit model.

3 A Simple Unit Model

Let us consider the reliability model of a simple unit, subject to gradual failures. These failures transfer the unit from one state to another, and these states are under control. It means that the failure can be found and repaired in such a way that after any repair the unit is returned back to the initial state. Suppose exponentially distributed times for any transition. Then the behavior of the unit is represented by a Markov process $X = \{X(t) : \ t \geq 0\}$ with discrete states space $E = \{0, 1, \ldots, n\}$ and transition intensities

$$a(x, y) = \begin{cases} \lambda(x) & \text{for } y = x + 1, \\ \mu(x) & \text{for } y = 0. \end{cases}$$

The Kolmogorov's system of differential equations for this process gets the form

$$\begin{aligned} \frac{d\pi(0;t)}{dt} + \lambda(0)\pi(0) &= \sum_{x \in E} \mu(x)\pi(x), \\ \frac{d\pi(x;t)}{dt} + \gamma(x)\pi(x) &= \lambda(x-1)\pi(x-1), \qquad x \in \{1, 2, \ldots, n\}. \end{aligned} \tag{3}$$

The system of complete balance equations for stationary probabilities of such a process can be obtained by eliminating the derivatives from equations (3). It should be noticed that the detailed balance equations do not take place for this model, and product form representation of steady state probabilities is not possible (Serfozo, 1999). Nevertheless, these equations admit some general solution, represented in the following theorem.

Theorem 1. *The steady state probabilities of the MRP of the Simple Unit Model have the form*

$$\pi(x) = g(x) \cdot \left[1 + \sum_{1 \leq x \leq n} g(x) \right]^{-1} \qquad \textit{with} \qquad g(x) = \prod_{1 \leq i \leq x} \frac{\lambda(i-1)}{\gamma(i)} \quad \textit{and } \ g(0) = 1. \tag{4}$$

A **proof** can be obtained by substitution of these expressions for the solution into the equations for the stationary probabilities, which are derived from (3). □

To calculate the reliability function of the simple unit with graduate controllable failures one should find the cumulative probability distribution function of the time to first entrance in the state n for process $X(t)$. For this purpose one should solve the system of equations (3) for the process with absorbing state n. By setting $\mu(n) = 0$ and denoting $\gamma(x; s) = s + \gamma(x)$ and $g(x; s) = \prod_{1\leq i\leq x} \lambda(i-1)\gamma(i; s)^{-1}$ the solution of this system can be found in terms of Laplace transform.

Theorem 2. *The reliability function of the simple unit with gradual failures is*

$$R(t) = 1 - \pi(n; t),$$

where $\pi(n; t)$ *defined by the Laplace transform*

$$\tilde{\pi}(n; s) = g(n; s) \left[\gamma(0; s) - \sum_{1\leq x\leq n} \mu(x) g(x; s) \right]^{-1}. \tag{5}$$

Proof Apply the Laplace transform to both sides of equations (3) with initial condition $\pi(0; 0) = \mathbf{P}\{X(0) = 0\} = 1$. Obtain equivalent algebraic system

$$\begin{aligned} (s + \lambda(0))\tilde{\pi}(0; s) - \sum_{1\leq x\leq n} \mu(x)\tilde{\pi}(x; s) &= 1, \\ (s + \gamma(x))\tilde{\pi}(x; s) - \lambda(x-1)\tilde{\pi}(x-1; s) &= 0, \qquad x \in \{1, 2, \ldots, n\}. \end{aligned}$$

Verify that formula (5) provides solution to these equations. □

4 Solution for a General Model

In terms of notations (1,2) the Kolmogorov's system of differential equations for the process state probabilities with transition intensities (1) and $\mu(\mathbf{x}) = \sum_{k\in\mathcal{K}} \mu_k(\mathbf{x})$ gets the form

$$\begin{aligned} \frac{d\pi(\mathbf{0}; t)}{dt} + \lambda(\mathbf{0})\pi(\mathbf{0}; t) &= \sum_{x\in E} \mu(\mathbf{x})\pi(\mathbf{x}), \\ \frac{d\pi(\mathbf{x}; t)}{dt} + \gamma(\mathbf{x})\pi(\mathbf{x}; t) &= \sum_{k\in\mathcal{K}} \lambda_k(\mathbf{x} - \mathbf{e}_k)\pi(\mathbf{x} - \mathbf{e}_k), \qquad \mathbf{x} \in E \setminus \{\mathbf{0}\}. \end{aligned} \tag{6}$$

The equations for stationary probabilities of the process can be obtained by eliminating derivatives (setting them equal to 0) in left side of (6), i.e. by taking the limit in both sides when t *rightarrow* ∞. Their solution can be represented in closed form. To find it let us denote by $p = (\mathbf{0} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n = \mathbf{x})$ some path from the state $\mathbf{x}_0 = \mathbf{0}$ to the state $\mathbf{x}_n = \mathbf{x}$, by $P(\mathbf{x})$ the set of all monotone paths from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}$. Let $\alpha(i)$e the coordinate (label of the unit) which should be changed in order to ensure the transition from state $\mathbf{x}_{i-1}$ to the state $\mathbf{x}_i$. Denote by $g(p)$ the following function along this path

$$g(p) = g(\mathbf{0} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n = \mathbf{x}) = \prod_{1\leq i\leq n} \frac{\lambda_{\alpha(i)}(\mathbf{x}_{i-1})}{\gamma(\mathbf{x}_i)},$$

and by $G(\mathbf{x})$ the sum of these functions along all the paths from state $\mathbf{0}$ to the state $\mathbf{x}$, namely

$$G(\mathbf{x}) \sum_{p\in P(\mathbf{x})} g(p) \quad \text{with} \quad G(\mathbf{0}) = 1.$$

Theorem 3. *The steady state probabilities of the MRP have the form*

$$\pi(\mathbf{x}) = \left[\sum_{x\in E} \sum_{p\in P(x)} g(p) \right]^{-1} G(\mathbf{x}), \quad \mathbf{x} \in E. \tag{7}$$

The **Proof** could be verified by substituting this form of the solution into the respective system of balance equations. □

Corollary. *The failure probability of the system π_F equals to the sum of the probabilities of the states over all failure set,*

$$\pi_F = \sum_{x \in F} \pi(\mathbf{x}).$$

Especially, if the failure set of the system reduces to only one of its units, then the failure function is $\pi_F = \pi_\Gamma$.

The reliability function of the system can be found as the distribution of the time to first entrance of the process $X(t)$ into the failure set F. Consider here the case, when $F = \Gamma$. The distribution of the time to first entrance of $X(t)$ into the set Γ can be found by solving the system (6) where any boundary state $\mathbf{x} \in \Gamma$ is considered as an absorbing state. The use of Laplace transforms simplifies the solution of system (6), where $\mu(\mathbf{x}) = 0$ is used for all boundary states $\mathbf{x} \in \Gamma$. System (6) turns into the system of algebraic equations

$$\begin{aligned} (s + \lambda(\mathbf{0}))\tilde{\pi}(\mathbf{0}; s) - \sum_{\mathbf{x} \in E} \mu(\mathbf{x})\tilde{\pi}(\mathbf{x}; s) &= 1 \\ (s + \gamma(\mathbf{x}, s)\tilde{\pi}(\mathbf{x}; s) - \sum_{k \in \mathcal{K}} \lambda_k(\mathbf{x} - \mathbf{e}_k)\tilde{\pi}(\mathbf{x} - \mathbf{e}_k; s) &= 0 \qquad \mathbf{x} \in E \setminus \mathbf{0}. \end{aligned} \tag{8}$$

To represent this solution denoting as before $\gamma(\mathbf{x}, s) = s + \gamma(\mathbf{x})$, and

$$g(p; s) = g((\mathbf{0} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n = \mathbf{x}); s) = \prod_{1 \le i \le n} \frac{\lambda_{\alpha(i)}(\mathbf{x}_{i-1})}{\gamma(\mathbf{x}_i; s)}, \text{ with } G(\mathbf{x}; s) = \sum_{p \in P(\mathbf{x})} g(p; s), \quad G(\mathbf{0}) = 1.$$

Theorem 4. *The reliability function of the considered system with gradual failures is*

$$R_S(t) = 1 - \pi_\Gamma(t),$$

where $\pi_\Gamma(t)$ is the distribution of the first entrance of $X(t)$ in the set Γ. It is given by $\pi_\Gamma(t) = \prod_{\mathbf{x} \in \Gamma} \pi(\mathbf{x}; t)$, with $\pi(\mathbf{x}; t)$ defined by theirs Laplace transforms

$$\tilde{\pi}(\mathbf{x}; s) = \left[\gamma(\mathbf{0}; s) - \sum_{\mathbf{x} \in E \setminus \mathbf{0}} \mu(\mathbf{x}) G(\mathbf{x}; s) \right]^{-1} G(\mathbf{x}; s), \;\; \mathbf{x} \in E. \tag{9}$$

The **proof** can be verified by substitution of this form of the solution into system (8). □

5 Conclusion

A general approach for studying reliability of hierarchical systems with gradual failures are represented in this paper. Special computer tools need to be developed for calculation of the reliability characteristics of real systems. Nevertheless some simple examples illustrate the power of this approach.

References

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