Stochastic Ordering of Order Statistics

Moshe Shaked
Department of Mathematics
University of Arizona
Tucson, Arizona 85721
U. S. A.
shaked@math.arizona.edu

Abstract

This is a survey of recent results involving comparisons of order statistics in the sense of various stochastic orders.

1 Motivation

Let $x_1, x_2, \ldots$ be a set of constants. Denote by $x_{(i:m)}$ the ith smallest value among the first m x_i's. Then we have the following inequalities (Mi and Shaked, 2001):

Lemma 1. *For any sequence of constants $x_1, x_2, \ldots$, the following inequalities hold:*

$$x_{(i:m)} \le x_{(i+1:m)}, \quad 1 \le i \le m-1. \tag{1}$$

$$x_{(i:m+1)} \le x_{(i:m)}, \quad 1 \le i \le m. \tag{2}$$

$$x_{(i:m)} \le x_{(i+1:m+1)}, \quad 1 \le i \le m. \tag{3}$$

Proof. The proof of (1) is obvious from the definition of the $x_{(i:m)}$'s. The proof of (2) is also quite simple: just note that if $x_{m+1} \le x_{(i:m)}$ then $x_{(i:m+1)} \le x_{(i:m)}$, whereas if $x_{m+1} > x_{(i:m)}$ then $x_{(i:m+1)} = x_{(i:m)}$. Finally, in order to prove (3), note that if $x_{m+1} \le x_{(i:m)}$ then $x_{(i+1:m+1)} = x_{(i:m)}$, whereas if $x_{m+1} > x_{(i:m)}$ then $x_{(i:m)} \le x_{(i+1:m+1)}$. □

Note that (1), (2), and (3) together can be written equivalently at follows:

Lemma 2. *For any sequence of constants $x_1, x_2, \ldots$, the following inequalities hold:*

$$x_{(i:m)} \le x_{(j:n)} \text{ whenever } i \le j \text{ and } m-i \ge n-j. \tag{4}$$

Proof. The inequalities (1), (2), and (3) easily follow from (4) by a proper choice of i, j, m and n. To show the converse assume that (1), (2), and (3) hold. If $m \ge n$ then

$$x_{(i:m)} \le x_{(i:n)} \le x_{(j:n)}$$

where the first inequality follows from (2) and $m \ge n$, and the second inequality follows from (1) and $i \le j$. And if $m < n$ then

$$x_{(i:m)} \le x_{(i+n-m:n)} \le x_{(j:n)}$$

where here the first inequality follows from (3) and $m < n$, and the second inequality follows from (1) and $j \ge i+n-m$. □

Now let $y_1, y_2, \ldots$ be another set of constants, and denote by $y_{(j:n)}$ the jth smallest value among the first n y_j's. Then (4) can be generalized as follows:

Lemma 3. *For any sequences of constants $x_1, x_2, \ldots$, and $y_1, y_2, \ldots$, if $x_i \le y_i$ for all i, then*

$$x_{(i:m)} \le y_{(j:n)} \text{ whenever } i \le j \text{ and } m - i \ge n - j. \tag{5}$$

Proof. Notice that the assumption yields $x_{(j:n)} \le y_{(j:n)}$. The stated result now follows from (4). □

In my talk I will describe some recent extensions of (4) and of (5) where the constants are replaced by random variables. Below, "increasing" stands for "non-decreasing."

2 The Usual Stochastic Order

A random vector $(X_1, X_2, \ldots, X_k)$ is said to be smaller than the random vector $(Y_1, Y_2, \ldots, Y_k)$ in the usual stochastic order (denoted by $(X_1, X_2, \ldots, X_k) \le_{\text{st}} (Y_1, Y_2, \ldots, Y_k)$) if $E[\phi(X_1, X_2, \ldots, X_k)] \le E[\phi(Y_1, Y_2, \ldots, Y_k)]$ for all componentwise nondecreasing functions ϕ for which the above expectations exist. The following extension of (5), for random variables, is taken from Mi and Shaked (2001).

Theorem 1. *Let $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$ be two sequences of random variables such that*

$$(X_1, X_2, \ldots, X_k) \le_{\text{st}} (Y_1, Y_2, \ldots, Y_k), \quad k \ge 1. \tag{6}$$

Then

$$X_{(i:m)} \le_{\text{st}} Y_{(j:n)} \text{ whenever } i \le j \text{ and } m - i \ge n - j. \tag{7}$$

Proof. The proof is an extension of the proofs of Lemmas 1–3. First note that from (6) it follows that

$$X_{(i:m)} \le_{\text{st}} Y_{(i:m)}, \quad 1 \le i \le m. \tag{8}$$

Now, if $m \ge n$ then

$$\begin{aligned} X_{(i:m)} &\le_{\text{a.s.}} X_{(i:n)} && \text{(by (2) and } m \ge n) \\ &\le_{\text{st}} Y_{(i:n)} && \text{(by (8))} \\ &\le_{\text{a.s.}} Y_{(j:n)} && \text{(by (1) and } i \le j). \end{aligned}$$

And if $m < n$ then

$$\begin{aligned} X_{(i:m)} &\le_{\text{st}} Y_{(i:m)} && \text{(by (8))} \\ &\le_{\text{a.s.}} Y_{(i+n-m:n)} && \text{(by (3) and } m < n) \\ &\le_{\text{a.s.}} Y_{(j:n)} && \text{(by (1) and } j \ge i + n - m). \end{aligned}$$

Since the almost sure relation $\le_{\text{a.s.}}$ implies the relation $\le_{\text{st}}$, we obtain (7) from the above inequalities. □

Note that if $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$ are each a sequence of *independent* random variables, such that $X_i \le_{\text{st}} Y_i$, for each i, then (6) holds. Therefore (7) holds. This significantly generalizes recent results of Nanda and Shaked (2001) and of Belzunce, Franco, Ruiz, and Ruiz (2001). In fact, in addition to the independence assumption, Nanda and Shaked (2001) had the additional assumption that all the above random variables are absolutely continuous, whereas Belzunce, Franco, Ruiz, and Ruiz (2001) proved it only for nonnegative random variables.

If in Theorem 1 we take $Y_i = X_i$, $i = 1, 2, \ldots$, then obviously (6) holds. Thus we obtain the following corollary, which is an extension of (4) for random variables:

Corollary 1. *Let $\{X_1, X_2, \ldots\}$ be a sequence of (not necessarily independent) random variables. Then*

$$X_{(i:m)} \le_{\text{st}} X_{(j:n)} \text{ whenever } i \le j \text{ and } m - i \ge n - j.$$

3 The Likelihood Ratio Order

Let X and Y be two absolutely continuous random variables with density functions f and g, respectively. Then X is said to be smaller than Y in the likelihood ratio order (denoted by $X \le_{\rm lr} Y$) if $g(t)/f(t)$ is increasing in t over the union of the supports of X and Y (here and below the convention $a/\infty = 0$ when $a > 0$ is used). Let $\overline{F}$ and $\overline{G}$ denote the survival functions of X and Y, respectively. Then X is said to be smaller than Y in the hazard rate order (denoted by $X \le_{\rm hr} Y$) if $\overline{G}(t)/\overline{F}(t)$ is increasing in t over $(-\infty, \max(u_X, u_Y))$ where u_X and u_Y are the right endpoints of the corresponding supports. Finally, let F and G denote the distribution functions of X and Y, respectively. Then X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \le_{\rm rh} Y$) if $G(t)/F(t)$ is increasing in t over $(\min(l_X, l_Y), \infty)$ where l_X and l_Y are the left endpoints of the corresponding supports. It is well known that $X \le_{\rm lr} Y \Longrightarrow X \le_{\rm hr} Y$ and that $X \le_{\rm lr} Y \Longrightarrow X \le_{\rm rh} Y$. Lillo, Nanda, and Shaked (2001) proved the following result:

Theorem 2. *Let $X_1, X_2, \ldots, X_m$ be m independent random variables, and let $Y_1, Y_2, \ldots, Y_n$ be other n independent random variables, all having absolutely continuous distributions. If* $\boxed{X_i \le_{\rm lr} Y_j}$ *for all $1 \le i \le m$ and $1 \le j \le n$, then*

$$X_{(i:m)} \le_{\rm lr} Y_{(j:n)} \text{ whenever } i \le j \text{ and } m - i \ge n - j. \tag{9}$$

Proof. Let f_i, F_i and $\overline{F}_i$ denote the density, distribution, and survival functions of X_i. Similarly, let g_j, G_j and $\overline{G}_j$ denote the density, distribution, and survival functions of Y_j. The ratio of the density functions of $Y_{(i:m)}$ and $X_{(j:n)}$ is given by

$$\frac{g_{Y_{(j:n)}}(t)}{f_{X_{(i:m)}}(t)} = \frac{\sum_{\boldsymbol{\sigma}} g_{\sigma_1}(t) G_{\sigma_2}(t) \cdots G_{\sigma_j}(t) \overline{G}_{\sigma_{j+1}}(t) \cdots \overline{G}_{\sigma_n}(t)}{\sum_{\boldsymbol{\pi}} f_{\pi_1}(t) F_{\pi_2}(t) \cdots F_{\pi_i}(t) \overline{F}_{\pi_{i+1}}(t) \cdots \overline{F}_{\pi_m}(t)}, \tag{10}$$

where $\sum_{\boldsymbol{\pi}}$ signifies the sum over all permutations $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ of $(1, 2, \ldots, m)$, and $\sum_{\boldsymbol{\sigma}}$ similarly denotes the sum over all permutations $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $(1, 2, \ldots, n)$. Now, for any choice of a permutation $\boldsymbol{\pi}$ of $(1, 2, \ldots, m)$ and a permutation $\boldsymbol{\sigma}$ of $(1, 2, \ldots, n)$ we have

$$\frac{g_{\sigma_1}(t) G_{\sigma_2}(t) \cdots G_{\sigma_j}(t) \overline{G}_{\sigma_{j+1}}(t) \cdots \overline{G}_{\sigma_n}(t)}{f_{\pi_1}(t) F_{\pi_2}(t) \cdots F_{\pi_i}(t) \overline{F}_{\pi_{i+1}}(t) \cdots \overline{F}_{\pi_m}(t)} = \frac{g_{\sigma_1}(t)}{f_{\pi_1}(t)} \cdot \frac{G_{\sigma_2}(t) \cdots G_{\sigma_i}(t)}{F_{\pi_2}(t) \cdots F_{\pi_i}(t)} \cdot \frac{\overline{G}_{\sigma_{j+1}}(t) \cdots \overline{G}_{\sigma_n}(t)}{\overline{F}_{\pi_{m-n+j+1}}(t) \cdots \overline{F}_{\pi_m}(t)} \cdot \frac{G_{\sigma_{i+1}}(t) \cdots G_{\sigma_j}(t)}{\overline{F}_{\pi_{i+1}}(t) \cdots \overline{F}_{\pi_{m-n+j}}(t)}.$$

Since $X_{\pi_1} \le_{\rm lr} Y_{\sigma_1}$ we see that the first fraction above is increasing in t. From $X_{\pi_k} \le_{\rm lr} Y_{\sigma_k}$ it follows that $X_{\pi_k} \le_{\rm rh} Y_{\sigma_k}$, but that means that $G_{\sigma_k}(t)/F_{\pi_k}(t)$ is increasing in t, $k = 2, \ldots, i$, and therefore the second fraction above is increasing in t. From $X_{\pi_{k+m-n}} \le_{\rm lr} Y_{\sigma_k}$ it follows that $X_{\pi_{k+m-n}} \le_{\rm hr} Y_{\sigma_k}$ but that means that $\overline{G}_{\sigma_k}(t)/\overline{F}_{\pi_{k+m-n}}(t)$ is increasing in t, $k = j+1, \ldots, n$, and therefore the third fraction above is increasing in t. The fourth fraction above obviously increases in t too, and thus the whole product increases in t.

Note that if $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ are all nonnegative univariate functions, such that $a_i(t)/b_j(t)$ is increasing in t for all $1 \le i \le m$ and $1 \le j \le n$, then $\sum_{i=1}^m a_i(t)/\sum_{j=1}^n b_j(t)$ is also increasing in t. It follows from this fact, and from (10), that $g_{Y_{(j:n)}}(t)/f_{X_{(i:m)}}(t)$ is increasing in t, and this gives the stated result. □

As a corollary of Theorem 2 we obtain that if $\{X_1, X_2, \ldots\}$ is a sequence of independent and identically distributed random variables, then

$$X_{(i:m)} \le_{\rm lr} X_{(j:n)} \text{ whenever } i \le j \text{ and } m - i \ge n - j.$$

4 The Hazard Rate and the Reversed Hazard Rate Orders

Boland and Proschan (1994) proved the following result:

Proposition 1. *Let $X_1, X_2, \ldots, X_m$ [respectively, $Y_1, Y_2, \ldots, Y_m$] be m independent (not necessarily i.i.d.) absolutely continuous random variables, all with support (a, b) for some $a < b$. If $X_i \le_{\rm hr} Y_j$ for all i and j, then $X_{(k:m)} \le_{\rm hr} Y_{(k:m)}$, $k = 1, 2, \ldots, m$.*

Using Proposition 1, Boland, Hu, Shaked, and Shanthikumar (2002) proved the following "hazard rate" analog of (9):

Theorem 3. *Let $X_1, X_2, \ldots, X_m$ be m independent (not necessarily identically distributed) random variables, and let $Y_1, Y_2, \ldots, Y_n$ be other n independent (not necessarily identically distributed) random variables, all having absolutely continuous distributions with support (a, b) for some $a < b$. If $\boxed{X_i \le_{\rm hr} Y_j}$ for all $1 \le i \le m$ and $1 \le j \le n$, then*

$$X_{(i:m)} \le_{\rm hr} Y_{(j:n)} \text{ whenever } i \le j \text{ and } m - i \ge n - j.$$

Proof. Let f_i and $\overline{F}_i$ denote the density and the survival functions of X_i. Similarly, let g_j and $\overline{G}_j$ denote the density and the survival functions of Y_j. Assume that $X_i \le_{\rm hr} Y_j$ for all i, j. First it will be shown that there exists a random variable Z with support (a, b) such that $X_i \le_{\rm hr} Z \le_{\rm hr} Y_j$ for all i, j. Let $r_{X_i} \equiv f_i/\overline{F}_i$ and $r_{Y_j} \equiv g_i/\overline{G}_i$ denote the hazard rate functions of the indicated random variables. From the assumption that $X_i \le_{\rm hr} Y_j$ for all i, j it follows that

$$\min\{r_{X_1}(t), r_{X_2}(t), \ldots, r_{X_m}(t)\} \ge \max\{r_{Y_1}(t), r_{Y_2}(t), \ldots, r_{Y_n}(t)\} \quad \text{for all } t \in (a, b).$$

Let q be a function which satisfies

$$\min\{r_{X_1}(t), r_{X_2}(t), \ldots, r_{X_m}(t)\} \ge q(t) \ge \max\{r_{Y_1}(t), r_{Y_2}(t), \ldots, r_{Y_n}(t)\} \quad \text{for all } t \in (a, b);$$

for example, let $q(t) = \min\{r_{X_1}(t), r_{X_2}(t), \ldots, r_{X_m}(t)\}$. It can be shown that q is indeed a hazard rate function. Let Z be a random variable with the hazard rate function q. Then indeed $X_i \le_{\rm hr} Z \le_{\rm hr} Y_j$ for all i, j.

Now, let $Z_1, Z_2, \ldots, Z_{\max\{m,n\}}$ be independent random variables, all distributed as Z. Then

$$\begin{aligned} X_{(i:m)} &\le_{\rm hr} Z_{(i:m)} && \text{(by Proposition 1)} \\ &\le_{\rm lr} Z_{(j:n)} && \text{(by (9))} \\ &\le_{\rm hr} Y_{(j:n)} && \text{(by Proposition 1),} \end{aligned}$$

and the stated result follows from the fact that the likelihood ratio order implies the hazard rate order. □

A result like Theorem 3, but with the order $\le_{\rm rh}$ replacing $\le_{\rm hr}$, is also valid.

Belzunce, F., Franco, M., Ruiz, J.-M., and Ruiz, M. C. (2001). On partial orderings between coherent systems with different structures. *Probability in the Engineering and Informational Sciences 15*, 273–293.

Boland, P. J., Hu, T., Shaked, M., and Shanthikumar, J. G. (2002). Stochastic ordering of order statistics II. In M. Dror, P. L'Ecuyer and F. Szidarovszky (Eds.), *Modeling Uncertainty: An Examination of Stochastic Theory, Methods, and Applications*, pp. 607–623. Boston: Kluwer.

Boland, P. J. and Proschan, F. (1994). Stochastic order in system reliability theory. In M. Shaked and J. G. Shanthikumar (Eds.), *Stochastic Orders and Their Applications*, pp. 485–508. Boston: Academic Press.

Lillo, R. E., Nanda, A. K., and Shaked, M. (2001). Preservation of some likelihood ratio stochastic orders by order statistics. *Statistics and Probability Letters 51*, 111–119.

Mi, J. and Shaked, M. (2001). First Order Stochastic Comparison of Order Statistics. Technical Report, Department of Mathematics, University of Arizona.

Nanda, A. K. and Shaked, M. (2001). The hazard rate and the reversed hazard rate orders, with applications to order statistics. *Annals of the Institute of Statistical Mathematics 53*, 853–864.