

MaGIC 2005

Ustaaset, Norway, 14-18 February

TJB Lectures

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Abstract

Three distinct but related topics will be discussed. The first part, consisting of four lectures, will be on the use of exterior algebra in numerics. The particular example to be studied in detail is the numerical integration of ordinary differential equations which arise when ODEs are restricted to exterior algebra spaces. The second part, consisting of one lecture, will be on the use of geometric integration for the numerical simulation of breaking water waves. The third part, consisting of three lectures, will be on multi-symplectic numerics. Existing results on M-S numerics will be reviewed, and some new ideas for M-S numerics, based on generalizing the exterior algebra framework of part 1 to PDEs, will be discussed.

1 Exterior algebra and ordinary differential equations

Given a vector space V , there are a number of other vector spaces that can be built on it: the dual space V^* , the spaces of k -vectors $\bigwedge^k(V)$, and k -forms $\bigwedge^k(V^*)$, for $k = 0, \dots, n$. Given an ordinary differential equation on V , it is often of interest to numerically integrate the induced systems on $\bigwedge^k(V)$ or $\bigwedge^k(V^*)$. These lectures will discuss the implementation and integration of such equations. No previous knowledge of exterior algebra will be assumed. The basics of exterior algebra will be introduced, and then applied to ODEs. A central topic is the *Grassmannian*. The Grassmannians associated with k -dimensional subspaces of V will be introduced and then related numerical issues discussed, including numerical preservation of the Grassmannian, and transforming so that the Grassmannian is attracting, and the role of boundary conditions. Other topics include: the various operations such as wedge and interior product, the Hodge star operator with and without a metric; extracting a basis from a k -vector or k -form; and, smoothness of subspaces.

A motivating example is the numerical integration of ODE eigenvalue problems on the real line, which arise in the study of the stability of solitary waves. Examples from this class of ODEs, will be used for illustration, as well as examples

from hydrodynamic stability. Another issue that arises in the study of such systems is preservation of holomorphicity in the discretization. For example, the numerical integration along a path in the complex plane, while preserving analyticity. Some aspects of discrete holomorphic functions will be discussed, such as numerical analytic continuation, and some issues of geometric integration raised.

The above theory is for general vector spaces. Two topics of interest when the vector space is symplectic is computation of the *Maslov index*, and the role of *Krein signature* in numerics. These concepts will be introduced, and then some recent results on computing the Maslov index using exterior algebra will be presented, and examples of the role of Krein signature presented.

2 Numerical simulation of breaking waves

The widely used governing equations for modelling water waves are Hamiltonian. The Hamiltonian structure was first identified by ZAKHAROV (1968). However this formulation is valid only for a simple free surface and is therefore not useful for breaking waves. It is not well-known that BENJAMIN & OLVER (1982) proposed a Hamiltonian formulation for any free surface, and they developed a coordinate-free formulation. In this lecture this geometric Hamiltonian formulation will be introduced and the issues associated with geometric integration discussed. The symplectic structure is non-canonical and is similar to a Lie-Poisson structure. Existing geometric integration technology does not apply, and new ideas are needed for numerical preservation of this structure. No previous knowledge of water waves will be assumed. The emphasis will be on numerical evolution of surfaces with a re-parameterization symmetry.

3 Multi-symplectic numerics

These lectures will begin by giving an overview of the concept of multi-symplecticity. Then previous work on *multi-symplectic integrators* will be reviewed, with particular attention to the formulations proposed by MARSDEN, PATRICK & SHKOLLER and BRIDGES & REICH, and related follow-up work by ASCHER, BAI, CHEN, GUO, LIU, MCLACHLAN, QIN, SCHÖBER, WANG, YANG, ZHEN and others.

A new approach to multi-symplectic numerics will also be introduced. On the *total exterior algebra* of a Riemannian manifold there is a canonical coordinate-free multi-symplectic structure. This structure can be constructed independent of any differential equation, but is also a natural model for a class of elliptic and hyperbolic PDEs. Numerical implementation of this formulation will be presented. It is a generalization to PDEs of the numerical exterior algebra framework presented in Part 1.