Local solutions to high frequency 3D scattering problems

Magic 2008, Renon, Italia
Andreas Asheim

in collaboration with Daan Huybrechs
NTNU Trondheim / KU Leuven

NTNU Norwegian University of Science and Technology
Outline

- Introduction: Scattering problems
- High frequency scattering
- Partial solutions to high frequency scattering problems
- Some numerical results
- Computing moments
Partial solutions to high frequency 3D scattering problems

Scattering problems

• Here: Scattering of waves
• Acoustic waves
  ➢ Seismic exploration
  ➢ Design
• Electromagnetic waves
  ➢ Radar
  ➢ Antenna design
Scattering problems

- Here: Scattering of waves
- Acoustic waves
  - Seismic exploration
  - Design
- Electromagnetic waves
  - Radar
  - Antenna design
- Reduces to the Helmholtz equation:
  \[ \Delta u + k^2 u = 0 \]
- Consider the exterior domain(∞)
Incoming field
Partial solutions to high frequency 3D scattering problems

Scattered field

$\mathcal{U}_S$
Partial solutions to high frequency 3D scattering problems

Combined field
Integral equation formulation

\[
\Delta u^i + k^2 u^i = 0 \quad \text{in} \quad \mathbb{R}^3 \\
\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega \\
u^s = -u^i \quad \text{on} \quad \partial \Omega
\]
Integral equation formulation

\[ \Delta u^i + k^2 u^i = 0 \quad \text{in} \quad \mathbb{R}^3 \]
\[ \Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega \]
\[ u^s = -u^i \quad \text{on} \quad \partial \Omega \]

Single layer potential:

\[ u^s(x) = \int_{\partial \Omega} K(x, y) q(y) ds_y \quad x \in \mathbb{R}^3 \setminus \Omega \]

\[ K(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \]
Partial solutions to high frequency 3D scattering problems

**Integral equation formulation**

\[
\begin{align*}
\Delta u^i + k^2 u^i &= 0 \quad \text{in} \quad \mathbb{R}^3 \\
\Delta u^s + k^2 u^s &= 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega \\
u^s &= -u^i \quad \text{on} \quad \partial \Omega
\end{align*}
\]

Single layer potential:

\[
u^s(x) = \int_{\partial \Omega} K(x, y) q(y) \mathrm{d}s_y \quad x \in \mathbb{R}^3 \setminus \Omega
\]

\[
K(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}
\]

Single layer potential density:

\[
\int_{\partial \Omega} K(x, y) q(y) \mathrm{d}s_y = u^i(x) \quad x \in \partial \Omega
\]
High frequency problems

\[ \int_{\partial \Omega} \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y) \, ds_y = u^i(x) \quad x \in \partial \Omega \]

- High frequency (large \( k \)):
  - \( q \) oscillates like \( u_i \) in illuminated region
  - \( q \) is small in shadow region
  - \( ? \) near shadow boundaries

...
Partial solutions to high frequency 3D scattering problems

**High frequency problems**

\[
\int_{\partial \Omega} \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y) \, ds_y = u^i(x) \quad x \in \partial \Omega
\]

- **High frequency (large \( k \)):**
  - \( q \) oscillates like \( u_i \) in illuminated region
  - \( q \) is small in shadow region
  - \( q \) is small in shadow region
  - \( q \) is small in shadow region near shadow boundaries

- **Classical numerical methods break down**
High frequency problems

\[ \int_{\partial \Omega} \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y) \, ds_y = u^i(x) \quad x \in \partial \Omega \]

- High frequency (large $k$):
  - $q$ oscillates like $u_i$ in illuminated region
  - $q$ is small in shadow region
  - $q$ near shadow boundaries
- Classical numerical methods break down
- Asymptotic expansions are available away from shadow boundary regions

Shadow boundaries
High frequency problems

\[ \int_{\partial \Omega} \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y) \, ds_y = u^i(x) \quad x \in \partial \Omega \]

- Phase of \( q \) obtained from asymptotic expansions

\[ q(y) = q_s(y)e^{ikg(y)} \]
High frequency problems

\[
\int_{\partial \Omega} \frac{e^{ik|x-y|}}{4\pi|x-y|} q(y) \, ds_y = u^i(x) \quad x \in \partial \Omega
\]

- Phase of \( q \) obtained from asymptotic expansions

\[
q(y) = q_s(y)e^{ikg(y)}
\]

- \( q_s \) will be smooth where the phase is well predicted
Partial solutions to high frequency 3D scattering problems

High frequency problems

$q$
Partial solutions to high frequency 3D scattering problems

High frequency problems

$q_s$
Partial solutions to high frequency 3D scattering problems

High frequency problems

- Solve for the smooth field $q_s$ instead of $q$
Partial solutions to high frequency 3D scattering problems

High frequency problems

- Solve for the smooth field $q_s$ instead of $q$
- Gives methods which do not deteriorate with increasing frequency
  - Huybrechs, Vandewalle (2006)
High frequency problems

\[ \int_{\partial \Omega} \frac{e^{ik(|x-y|+g_i(y))}}{4\pi|x-y|} q_s(y)\,ds_y = u_i(x) \quad x \in \partial \Omega \]
High frequency problems

\[
\int_{\partial \Omega} \frac{e^{ik(|x-y|+g_i(y))}}{4\pi|x-y|} q_s(y) \, ds \, y = u_i(x) \quad x \in \partial \Omega
\]

\[
\text{Error } \sim O(k^{-p})
\]

\[
\sum_{j=1}^{N} \sum_{\nu \leq m} w_{j,\nu} \partial^{\nu} q_s(\tau_j) = u_i(x) \quad x \in \partial \Omega
\]
Partial solutions to high frequency 3D scattering problems

High frequency problems

\[ \int_{\partial \Omega} \frac{e^{ik(|x-y|+g_i(y))}}{4\pi|x-y|} q_s(y) \, ds_y = u_i(x) \quad x \in \partial \Omega \]

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\[ \sum_{j=1}^{N} \sum_{\nu \leq m} w_{j,\nu} \partial^{\nu} q_s(\tau_j) = u_i(x) \quad x \in \partial \Omega \]

- **Filon-type**
  - Data from \( N \) contributing points (endpoints, stationary points, singularities)
  - Asymptotic accuracy
  - Accurate for moderate \( k \)
High frequency problems

\[
\int_{\partial \Omega} \frac{e^{ik(|x-y|+g_i(y))}}{4\pi |x-y|} q_s(y) \, ds_y = u_i(x) \quad x \in \partial \Omega
\]

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\text{Error} \sim \mathcal{O}(k^{-p})
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\sum_{j=1}^{N} \sum_{\nu \leq m} w_{j,\nu} \partial^\nu q_s(\tau_j) = u_i(x) \quad x \in \partial \Omega
\]

- Filon-type
  - Data from \( N \) contributing points (endpoints, stationary points, singularities)
  - Asymptotic accuracy
  - Accurate for moderate \( k \)

- Can show that \( x \) is the only contributing point
Partial solutions to high frequency 3D scattering problems

Partial solutions
Partial solutions to high frequency 3D scattering problems

Partial solutions

• Classical discretization gives dense matrices
• Classical physics states localization
  - high frequency,
  - illuminated region
Partial solutions

- Classical discretization gives dense matrices
- Classical physics states localization
  - high frequency,
  - illuminated region
- Filon approach for the high frequency problems is localized
  - Results in sparse discretization matrices (Huybrechs, Vandewalle 2006)
Partial solutions to high frequency 3D scattering problems

Partial solutions

- Classical discretization gives dense matrices
- Classical physics states localization
  - high frequency,
  - illuminated region
- Filon approach for the high frequency problems is localized
  - Results in sparse discretization matrices (Huybrechs, Vandewalle 2006)
- Idea: Solve only for parts of the illuminated region
  - As a part of a larger computation
  - Certain regions are more interesting
  - Multiple scattering
Partial solutions to high frequency 3D scattering problems

Example: Multiple scattering

- Results by Fatih Ecevit:
- Solve high frequency problems on non-convex configurations
Partial solutions to high frequency 3D scattering problems

Example: Multiple scattering

• Results by Fatih Ecevit:
  • Solve high frequency problems on non-convex configurations
    ➢ Split domain into convex parts
    ➢ Solve iteratively
    ➢ Extrapolate
Partial solutions to high frequency 3D scattering problems

Example: Multiple scattering

- Results by Fatih Ecevit:
- Solve high frequency problems on non-convex configurations
  - Split domain into convex parts
  - Solve iteratively
  - Extrapolate
- Computational complexity does not increase with $k$. 
Partial solutions to high frequency 3D scattering problems

**Example: Multiple scattering**

- Results by Fatih Ecevit:
- Solve high frequency problems on non-convex configurations
  - Split domain into convex parts
  - Solve iteratively
  - Extrapolate
- Computational complexity does not increase with $k$. 

![Diagram showing multiple scattering with domains $\Omega_1$ and $\Omega_2$]
Example: Multiple scattering

- Results by Fatih Ecevit:
- Solve high frequency problems on non-convex configurations
  - Split domain into convex parts
  - Solve iteratively
  - Extrapolate
- Computational complexity does not increase with $k$. 
Partial solutions to high frequency 3D scattering problems

Partial solutions:
Discretizing the smooth field

\[ \sum_{\nu \leq m} w_\nu \partial^\nu q_s(x) = u_i(x) \]

- Need \( q_s \) represented in a finite dimensional space

[Graph showing the real value of the smooth field]
Partial solutions to high frequency 3D scattering problems

Partial solutions:
Discretizing the smooth field

\[ \sum_{\nu \leq m} w_{\nu} \partial^{\nu} q_s(x) = u(x) \]

- Need \( q_s \) represented in a finite dimensional space
- Natural splines, clamped B-spline surface
  
  \( \triangleright \) Order > \( m \)
Partial solutions to high frequency 3D scattering problems

Partial solutions: Discretizing the smooth field

\[ \sum_{\nu \leq m} w_\nu \partial^\nu q_s(x) = u_i(x) \]

- Need \( q_s \) represented in a finite dimensional space
- Natural splines, clamped B-spline surface
  - Order > \( m \)
- Results in sparse system
Partial solutions to high frequency 3D scattering problems

Partial Solutions:
2D experiment

\[ \sum_{j=0}^{m} w_j q_s^{(j)}(x) = u_i(x) \]

\[ \theta = \frac{3}{10} \pi \]

![Graph showing error as a function of \( k \) for different values of \( m \).](image)
Partial solutions to high frequency 3D scattering problems

Partial Solutions:
2D experiment

\[ \sum_{j=0}^{m} w_j q_s^{(j)}(x) = u_i(x) \]

\[ \theta = \frac{2}{5} \pi \]

\[ \text{Error} \]

\[ k \]

\[ m = 0 \]

\[ m = 1 \]

\[ m = 2 \]
Partial solutions to high frequency 3D scattering problems

Partial Solutions:
3D experiment
Computing moments

\[
\sum_{\nu \leq m} w_{\nu} \partial^\nu q_s(x) = u_i(x) \quad x \in \partial \Omega
\]

- The weights \( w_{\nu} \) are highly oscillatory integrals of the form

\[
\int \int \frac{e^{ik(d_{t_0}(t) + g(t))}}{d_{t_0}(t)} f(t) dt
\]
Computing moments

\[ \sum_{\nu \leq m} w_{\nu} \partial^\nu q_s(x) = u_i(x) \quad x \in \partial \Omega \]

- The weights \( w_{\nu} \) are highly oscillatory integrals of the form

\[ \int \int \frac{e^{ik(d_{t_0}(t)+g(t))}}{d_{t_0}(t)} f(t)dt \]

- Computed with numerical steepest descent
Partial solutions to high frequency 3D scattering problems

Computing moments

• **Numerical steepest descent method:**
  - Deform path from $a$ to $b$ into the complex plane

\[
\int_{a}^{b} f(x) e^{ikg(x)} \, dx
\]
Computing moments

• Numerical steepest descent method:
  - Deform path from $a$ to $b$ into the complex plane
  - Eliminate oscillations by following steepest descent path:
    \[ g(h_x(p)) = g(x) + ip \]
Partial solutions to high frequency 3D scattering problems

Computing moments

- Numerical steepest descent method:
  - Deform path from $a$ to $b$ into the complex plane
  - Eliminate oscillations by following steepest descent path:

\[
\int_{a}^{b} f(x)e^{ikg(x)} \, dx
\]

\[
g(h_x(p)) = g(x) + ip
\]

- Path must pass through all special points: Endpoints, singularities and stationary points
Computing moments

- **Numerical steepest descent method:**
  - Deform path from $a$ to $b$ into the complex plane
  - Eliminate oscillations by following steepest descent path:
    \[
    g(h_x(p)) = g(x) + ip
    \]
  - Path must pass through all special points: Endpoints, singularities and stationary points
  - Expresses the integral as a sum of contributions from special points
    \[
    \int_a^b f(x)e^{ikg(x)}\,dx = F(b) - F(a) + \ldots
    \]
  - Where
    \[
    F(x) = e^{ikg(x)} \int_0^\infty f(h_x(p))h'_x(p)e^{-kp}\,dp
    \]
Computing moments

• 2D Numerical steepest descent:
  ➢ Treated as nested 1D integrals

\[
\int_{a}^{b} \int_{c}^{d} f(x, y) e^{ikg(x,y)} \, dx \, dy
\]
Computing moments

- 2D Numerical steepest descent:
  - Treated as nested 1D integrals
  - Steepest descent path now depends on $y$

$$g(u_x(p, y), y) = g(x, y) + ip$$
Computing moments

- 2D Numerical steepest descent:
  - Treated as nested 1D integrals
  - Steepest descent path now depends on \( y \)
  - Outer integral as before, path \( v(q) \)
  - Expresses the integral as a sum of contributions from special points (corner points, stationary points, resonance points and singularities)

\[
\int_a^b \int_c^d f(x, y)e^{ikg(x, y)}\,dx\,dy = G(d, b) - G(c, b) - G(d, a) + G(a, c) + \ldots
\]

where
\[
G(x, y) = \int_0^\infty \int_0^\infty \tilde{f}_{x, y}(p, q)e^{-k(p+q)} \frac{\partial u}{\partial p} \frac{dv}{dq} \,dp\,dq
\]
Partial solutions to high frequency 3D scattering problems

Computing moments

- In our case the only contributing point is $x$ (periodic boundaries.)
Partial solutions to high frequency 3D scattering problems

Computing moments

- In our case the only contributing point is \( x \) (periodic boundaries.)
- This means only four contributing integrals of the form.

\[
\int_0^\infty \int_0^\infty \frac{e^{-k(p^n+q^m)}}{\sqrt{p^2 + q^2}} \varphi(p, q) \, dp \, dq
\]
Partial solutions to high frequency 3D scattering problems

Computing moments

- In our case the only contributing point is \( x \) (periodic boundaries.)
- This means only four contributing integrals of the form.

\[
\int_0^\infty \int_0^\infty \frac{e^{-\frac{k}{\sqrt{p^2 + q^2}}} \varphi(p, q)dpdq}{\sqrt{p^2 + q^2}}
\]

- Computed with generic software
  - Polar coordinates eliminates the singularity
Partial solutions to high frequency 3D scattering problems

Technical comments

- When the path cannot be computed exactly, an approximation will often be sufficient.
  - Taylor expansions
  - Newton-Rhapson iterations
Partial solutions to high frequency 3D scattering problems

Technical comments

• When the path cannot be computed exactly, an approximation will often be sufficient.
  ➢ Taylor expansions
  ➢ Newton-Rhapson iterations

• Main problem: Need analytical continuations of real functions
  ➢ Branch cut of $\sqrt{}$ must be moved.
  ➢ Must be able to predict singular behaviours.
Open question

- Balancing errors
  - Discretization
  - Quadrature
  - Convergence?

- Conditions for high asymptotic order
Thank you