Shape analysis using geodesic paths on shape manifolds

Eivind Fonn

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What are shapes?

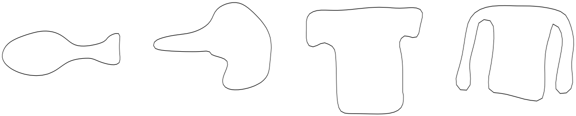

Figure: Please excuse my artistic attempts. . .

Closed curves in 2D.

But what *are* they?

In this context, equivalence classes of parametrized curves $\gamma : S^1 \to \mathbb{R}^2$ satisfying certain conditions (*preshapes*). Assume γ is differentiable and write

$$\gamma'(t) = \exp\left[\nu(t) + i\theta(t)\right].$$

We represent the preshape γ by the pair (ν, θ).

Conditions?

The functions ν and θ must satisfy

- invariance under scaling $(\nu \to \nu + c)$, so require

$$\int_0^{2\pi} e^{\nu(t)}\,\mathrm{d}t = 2\pi \qquad \textit{or} \qquad \nu(t) \equiv 0.$$

- invariance under rotation $(\theta \to \theta + c)$, so require

$$\int_0^{2\pi} e^{\nu(t)}\theta(t)\,\mathrm{d}t = 0 \qquad \textit{or} \qquad \int_0^{2\pi} \theta(t)\,\mathrm{d}t = 0.$$

More conditions?

You bet. We need to ensure that (ν, θ) represents a closed curve, so we require that

$$\int_0^{2\pi} e^{\nu(t)} \sin\theta(t)\,\mathrm{d}t = \int_0^{2\pi} e^{\nu(t)} \cos\theta(t)\,\mathrm{d}t = 0,$$

or

$$\int_0^{2\pi} \sin\theta(t)\,\mathrm{d}t = \int_0^{2\pi} \cos\theta(t)\,\mathrm{d}t = 0.$$

Also useful is the index condition,

$$\theta(t + 2\pi) = \theta(t) + 2\pi.$$

Preshape spaces

The inelastic preshape space

$$C_i = \left\{ \theta \in L^2_{sp} \;\middle|\; \int \sin\theta = \int \cos\theta = \int \theta = 0 \right\}.$$

The elastic preshape space

$$C_e = \left\{ (\nu, \theta) \in L^2_p \times L^2_{sp} \;\middle|\; \int e^{\nu} = 2\pi, \right.$$
$$\left. \int e^{\nu} \sin\theta = \int e^{\nu} \cos\theta = \int e^{\nu}\theta = 0 \right\}.$$

Shape spaces

The inelastic shape space

$$S_i = C_i / S^1.$$

The elastic shape space

$$S_e = C_e / D(S^1).$$

Tools

Common spare-time activities include

- Riemannian metrics.
- Retractions.
- Levi-Civita connections.
- Geodesics . . . our main concern.

The shooting method

Find a geodesic from $\theta_0 \in C_i$ to $\theta_1 \in C_i$ by "shooting" in different directions and picking the one that gets you closest. Basically a minimization problem on $T_{\theta_0} C_i$.

Advantages:

- Simple to formulate.
- Geodesics with IC are easy to calculate.
- Easy to extend to finding geodesics in S_i.

Disadvantages:

- Objective function is a beast.
- Approximating gradients numerically is slooooow.

The "direct" method

Discretize the equations that *define* a geodesic on C_i, which leads to finding a zero of a nonlinear function on $\mathbb{R}^k$.

Advantages:

- Objective function in closed form, can calculate Jacobians analytically.
- Potentially faster convergence (Newton's method anybody?).

Disadvantages:

- Not so simple to formulate (but simple enough).
- Requires much more memory (but little enough).
- Harder to extend to finding geodesics in S_i.

The "direct" method

The continuous inelastic problem is: Find $\theta_s(t)$ for $(s,t) \in (0,1) \times (0,2\pi)$, given boundary conditions $\theta_0(t), \theta_1(t)$, such that θ_s lies on C_i,

$$\int \sin\theta_s = \int \cos\theta_s = \int \theta_s = 0 \text{ for all } s,$$

and is geodesic, there exists $a^1(s), a^2(s), a^3(s)$ such that

$$\partial^2\theta_s(t)/\partial s^2 = a^1(s)\sin\theta_s(t) + a^2(s)\cos\theta_s(t) + a^3(s).$$

The functions a^i are unknown.

Extending to shapes

Problem: Want to consider all reparametrizations of the final preshape θ_1.

- Shooting method: Easy. Include a “shape matcher” in the objective function.
- Direct method: Harder. Bruteforce or heuristic search.

Problem: There may be shorter geodesics.

- No solution. . .

Fish to duck

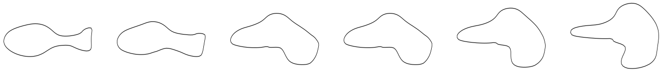

Figure: The shooting method.

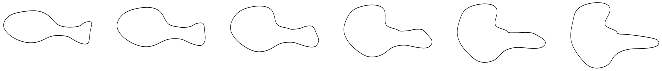

Figure: The direct method.

T-shirt to sweater

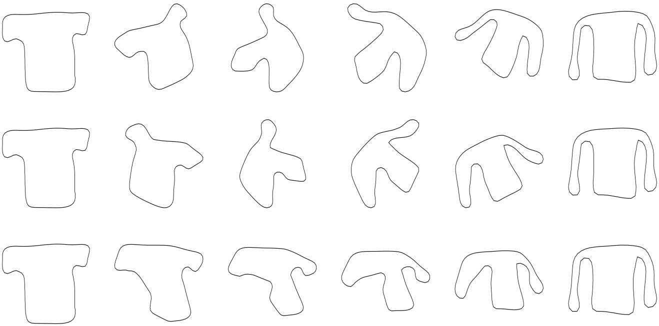

Figure: The shooting method can give various results depending on the initial guess.

T-shirt to sweater

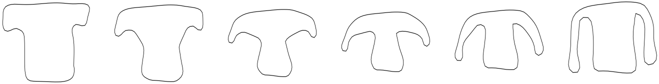

Figure: The direct method seems more robust, and the geodesics seem more natural.

Others

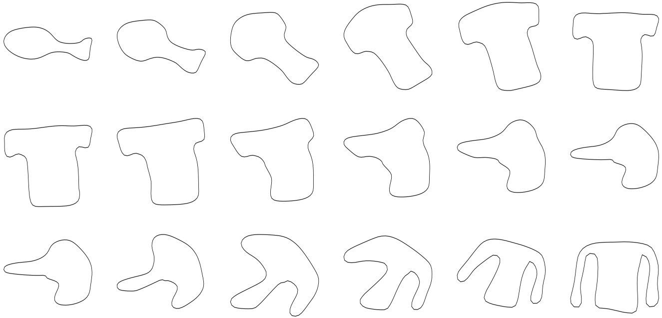

Figure: All calculated with the direct method.

Times

Geodesic	Shoot (20)	Direct (20)	Shoot (10)	Direct (10)
F→D	15.85/12.86	4.30	15.08/10.95	0.93
T→S	94.26/174.3	6.01	62.73/114.1	1.26
F→T	33.74/34.80	3.60	25.94/17.07	0.88
T→D	41.59/41.97	4.12	25.36/22.37	0.90
D→S	187.9/285.4	5.52	105.3/197.4	1.16
S→F	53.60/61.63	∞	42.22/40.50	∞

Table: How long the geodesics took to calculate, on my personal laptop. . .