Optimisation on Lie Groups Applied to Optical Interference Filters

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NTNU

Outline

1. Motivation
   - Picture Tour

Outline

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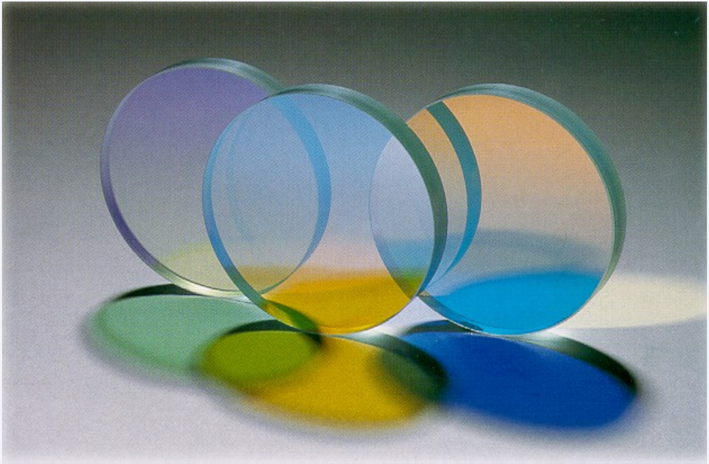
Some Pretty Pictures
Optical Interference Filters



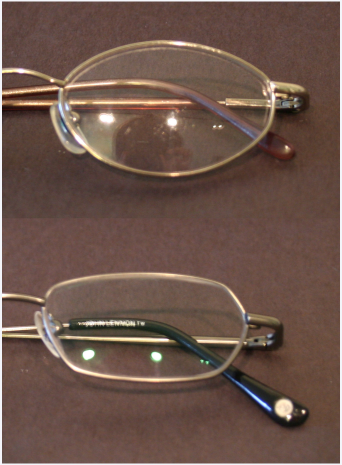
Source: Wikipedia

Some Pretty Pictures

Diesel Rainbow

Source: Wikipedia

Some Pretty Pictures
Anti-Reflective Coating



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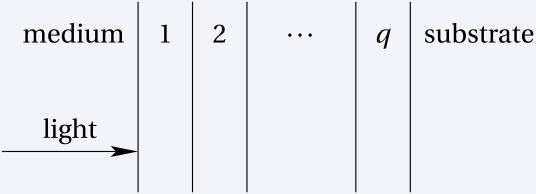
Layered Structure

An optical interference filter is layered.



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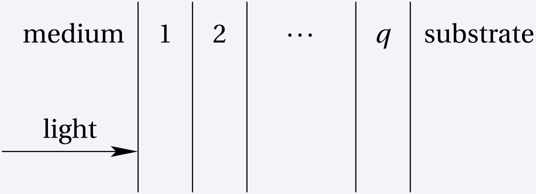

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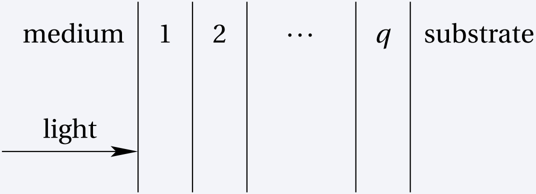

- Each layer corresponds to a matrix in $SL(2,\mathbb{C})$.
- The whole stack is characterised by the product of these matrices.

Goal

Design an optimal one-layer anti-reflective filter.

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Design an optimal one-layer anti-reflective filter.

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- The light is monochromatic.

Derivation of the Electromagnetic Wave Equation

We start from Maxwell's equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{D} = \rho,$$
$$\nabla \cdot \mathbf{B} = 0,$$

together with the relations

$$\mathbf{J} = \sigma \mathbf{E},$$
$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E},$$
$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mu_r \mathbf{H}.$$

Derivation of the Electromagnetic Wave Equation

We eliminate **J**, **B** and **D**,

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This gives us (under the previous assumptions)

$$-\nabla \times (\nabla \times \mathbf{E}) = \nabla^2 \mathbf{E} = \sigma\mu \frac{\partial \mathbf{E}}{\partial t} + \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2},$$
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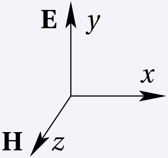
Sinusoidal Wave Solution



$$\mathbf{E} = E\hat{\mathbf{y}} = \mathscr{E}\mathrm{e}^{\mathrm{i}\omega(t-x/v)}\hat{\mathbf{y}}, \qquad \mathbf{H} = H\hat{\mathbf{z}} = \mathscr{H}\mathrm{e}^{\mathrm{i}\omega(t-x/v)}\hat{\mathbf{z}}$$

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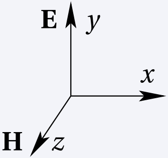

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Dispersion relation

Sinusoidal solution if

$$\frac{\omega^2}{v^2} = \epsilon\mu\omega^2 - \mathrm{i}\sigma\mu\omega$$

Sinusoidal Wave Solution

In vacuum, $\sigma = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$ and $v = c$. We then get

$$c^2 = \frac{1}{\epsilon_0 \mu_0}.$$

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Complex Refractive Index

$$N \stackrel{\text{def}}{=} n - \mathrm{i}k \stackrel{\text{def}}{=} \frac{c}{v}$$

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Letting $\lambda_0 \stackrel{\text{def}}{=} 2\pi c/\omega$,

$$\frac{E}{\mathscr{E}} = \frac{H}{\mathscr{H}} = \mathrm{e}^{\mathrm{i}(\omega t - (2\pi N/\lambda_0)x)} = \mathrm{e}^{-(2\pi k/\lambda_0)x} \mathrm{e}^{\mathrm{i}(\omega t - (2\pi n/\lambda_0)x)}.$$

Relationship Between the Electric and Magnetic Fields

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}, \quad \mathbf{E} = E\hat{\mathbf{y}} = \mathscr{E} \mathrm{e}^{\mathrm{i}\omega(t - x/v)}\hat{\mathbf{y}}, \quad \mathbf{H} = H\hat{\mathbf{z}} = \mathscr{H} \mathrm{e}^{\mathrm{i}\omega(t - x/v)}\hat{\mathbf{z}}$$

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$$-\frac{\partial H}{\partial x}\hat{\mathbf{y}} = (\sigma + \mathrm{i}\epsilon\omega) E\hat{\mathbf{y}}$$

$$\mathrm{i}\frac{\omega}{v} H = \mathrm{i}\frac{\omega N^2 \epsilon_0}{\mu_r} E$$

$$H = \frac{N}{c\mu} E$$

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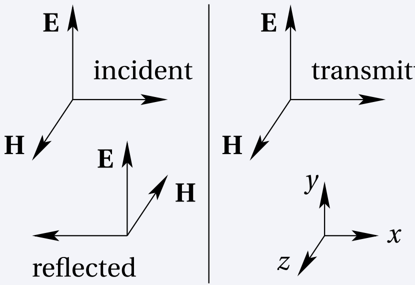
Characteristic Optical Admittance

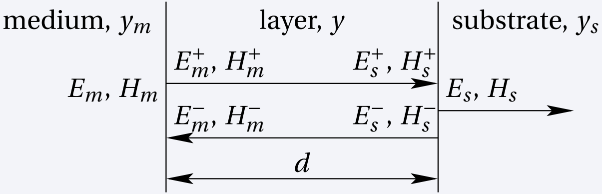
$$y \overset{\text{def}}{=} \frac{N}{c\mu} \quad \Longrightarrow \quad H = yE$$

What Happens at an Interface?



$$E_t = E_i + E_r, \qquad H_t = H_i - H_r.$$

Interference



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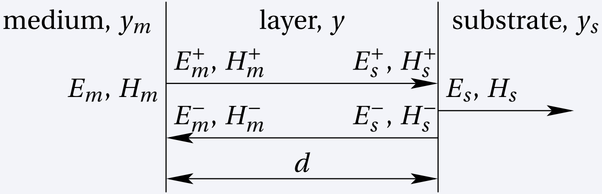
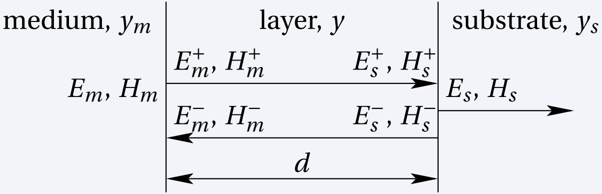


$$\delta \stackrel{\text{def}}{=} \frac{2\pi N d}{\lambda_0}$$

$$E_m^+ = E_s^+ \mathrm{e}^{\mathrm{i}\delta}, \qquad E_m^- = E_s^- \mathrm{e}^{-\mathrm{i}\delta}.$$

Interference



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$$\begin{bmatrix} E_m \\ H_m \end{bmatrix} = \begin{bmatrix} \cos\delta & (\mathrm{i}\sin\delta)/y \\ \mathrm{i}y\sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} E_s \\ H_s \end{bmatrix}$$

Generalisation to q Layers

$$\begin{bmatrix} E_m \\ H_m \end{bmatrix} = \left(\prod_{r=1}^{q} \begin{bmatrix} \cos\delta_r & (\mathrm{i}\sin\delta_r)/y_r \\ \mathrm{i}y_r\sin\delta_r & \cos\delta_r \end{bmatrix} \right) \begin{bmatrix} E_s \\ H_s \end{bmatrix}$$

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Normalised Version

$$\begin{bmatrix} B \\ C \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} E_m/E_s \\ H_m/E_s \end{bmatrix} = \left(\prod_{r=1}^{q} \begin{bmatrix} \cos\delta_r & (\mathrm{i}\sin\delta_r)/y_r \\ \mathrm{i}y_r\sin\delta_r & \cos\delta_r \end{bmatrix} \right) \begin{bmatrix} 1 \\ y_s \end{bmatrix}$$

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Reflectance

$$R = \left| \frac{y_m B - C}{y_m B + C} \right|^2$$

Parametrisation

$$R\colon \mathrm{SL}(2,\mathbb{C}) \to \mathbb{R}$$

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$$R\colon \mathrm{SL}(2,\mathbb{C}) \to \mathbb{R}$$

Parametrisation in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ which consists of trace-free matrices.

Basis of $\mathfrak{sl}(2,\mathbb{C})$

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Parametrisation

Suppose $V \in \mathfrak{sl}(2,\mathbb{C})$ and λ is an eigenvalue of V. Then

$$e^V = I\cosh\lambda + V\frac{\sinh\lambda}{\lambda} \in SL(2,\mathbb{C}).$$

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$$\mathrm{e}^{V} = I\cosh\lambda + V\frac{\sinh\lambda}{\lambda} \in \mathrm{SL}(2,\mathbb{C}).$$

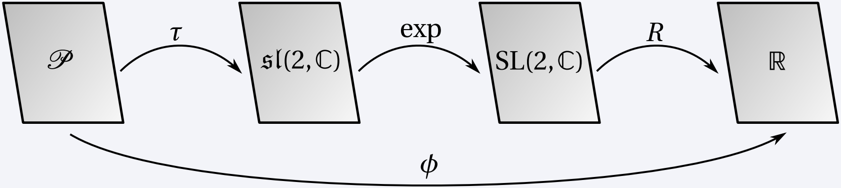
All physically realisable matrices in $\mathfrak{sl}(2,\mathbb{C})$ are matrices of the form

$$V = -\lambda\begin{bmatrix} 0 & 1/y \\ y & 0 \end{bmatrix},$$

or by using the basis,

$$= -\frac{\lambda}{y}E_2 - \lambda y E_3.$$

Parametrisation



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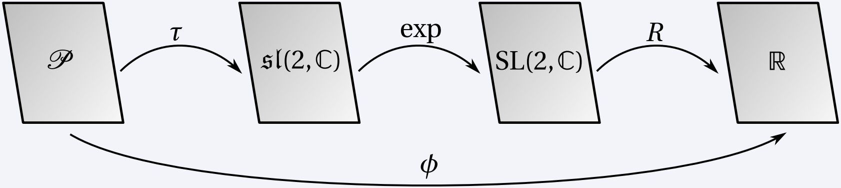

BFGS-minimisation of

$$\phi = R \circ \exp \circ \tau.$$

Future Work

- Multiple layers

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- Consider the whole spectrum

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- Continuously varying parameters

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- Multiple layers
- Consider the whole spectrum
- Continuously varying parameters
- Other optimisation algorithms

Further Reading

H. Angus Macleod.
Thin-Film Optical Filters, 3rd ed.
Institute of Physics Publishing, 2001.

Stéphane Larouche and Ludvik Martinu.
OpenFilters: Open-Source Software for the Design, Optimization, and Synthesis of Optical Filters.
Applied Optics 47, 219–230, 2008.