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Generalized affine groups in exponential integrators

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Most exponential integrators use functions similar to the exponential in their implementation. This report describes a structure in which these function play a central role, and the framework developed herein is believed to be applicable to further studies of these functions. The framework is constructed using ideas from Lie group methods, with generalized affine groups as the fundamental building block, for the solution of non-autonomous systems of ordinary differential equations. This work can be seen as an extension to work by Minchev [10].

1. Introduction

Exponential integrators are numerical schemes tailored for systems of ordinary differential equations of the type

$$y'(t) = Ly + N(y,t), \qquad y(t_0) = y(0) \tag{1}$$

in which L is a linear operator and $N(y,t)$ is a (possibly nonlinear) function. If, for instance, L is such that (1) is stiff, classical explicit Runge–Kutta integrators encounter step size restrictions, and the aim of exponential integrators is to enable the use of explicit schemes without step size restrictions.

A building block of exponential integrators is the "φ functions", which we define by the integral representation

$$\varphi_j(z) = \frac{1}{(j-1)!}\int_0^1 \mathrm{e}^{(\theta-1)z}\theta^{j-1}\,\mathrm{d}\theta, \qquad j = 1,2,\ldots, \tag{2}$$

which for $j = 1,2,3$ (and for $z \neq 0$) can be calculated as

$$\varphi_1(z) = \frac{\mathrm{e}^z - 1}{z}, \quad \varphi_2(z) = \frac{\mathrm{e}^z - z - 1}{z^2} \quad \text{and} \quad \varphi_3(z) = \frac{\mathrm{e}^z - z^2/2 - z - 1}{z^3}.$$

Numerous exponential integrators exist for the equation (1), we refer to [3] and its accompanying software for a list. Most of them make use of φ functions, but the Lawson integrators are examples that do not [8, 2].

Lie group integrators may be extended to exponential integrators through the affine group (of degree one) $\mathrm{GL}_n(\mathbf{R})\rtimes\mathbf{R}^n$ and its affine action on $\mathbf{R}^n$. This was already mentioned in [11]. The simplest affine Lie group integrator for (1) is then

$$y_{n+1} = \mathrm{e}^{hL}y_n + h\varphi_1(hL)N(y_n,t_n) \tag{3}$$

where $y_n \approx y(t_n)$. This scheme can be called Lie–Euler in our context, but is also known as Nørsett–Euler, ETD Euler, filtered Euler, exponentially fitted Euler etc.

The important feature of (3) in this context is the use of the function φ_1. ϕ functions (2) is a class of functions that are all exponential-like, but has not been as extensively studied in mathematical literature such as the exponential function itself.

The motivation for this work was to construct a group around the φ functions in order to reveal further properties. All details on these generalized affine groups are developed in Section 2. A direct application of Corollary 2.8 herein already exists for the implementation of the scaling and squaring approach of evaluating φ functions in [3].

The report [10] has the same framework up to degree $d = 2$, and focuses mainly on constructing Lie group integrators using the higher order affine groups. This report owes a lot to [10] and [7] for ideas and inspiration. Section 3 exemplifies how this framework can be applied for $d = 2$ in order to construct Lie group integrators.

2. Generalized affine groups and their Lie algebras

The well-known affine group, $\mathrm{GL}_n(\mathbf{R}) \rtimes \mathbf{R}^n$ and its corresponding group action on $\mathbf{R}^n$, $(A, b) \cdot y = Ay + b$, is in this section generalized to higher degree by incorporating further $\mathbf{R}^n$-vectors into the group. The basic affine group is denoted degree $d = 1$, as the group has one $\mathbf{R}^n$ vector. One perhaps unexpected necessity for higher order groups, is the introduction of an additional scalar, λ, which is needed for this framework to be able to solve non-autonomous differential equations. The parameter λ is used in the algebra to maintain a notion of unity time.

We start with Lemma 2.1 which is well known in recent texts on exponential integrators. φ functions appear in this lemma as a convenient symbol for a recurring expression. Then an affine algebra, an affine group, an exponential and logarithm connecting the algebra and the group is defined. Lemma 2.5 then proves that this framework is able to solve the differential equation (1).

Lemma 2.7 is an application of the group structure which gives meaning to φ functions on sums. It has the immediate Corollary 2.8 which is already useful for the current numerical implementation of φ functions, and also Corollary 2.9 which is yet to be applied for future backward error analysis of φ functions.

Lemma 2.1. *The nonautonomous differential equation*

$$y'(t) = \alpha y + \sum_{j=0}^{d-1} \frac{t^j}{j!}\beta^{[j]}, \qquad y(t_0) = y_0 \tag{4}$$

for $y(t) \in \mathbf{R}^n$, $\alpha \in \mathbf{R}^{n\times n}$, $\beta^{[j]} \in \mathbf{R}^n$, has the exact solution

$$y(t_0 + h) = \mathrm{e}^{h\alpha} y_0 + \sum_{j=0}^{d-1}\sum_{k=0}^{j} \frac{t_0^{j-k}}{(j-k)!} h^{k+1}\varphi_{k+1}(h\alpha)\beta^{[j]} \tag{5}$$

with φ_k defined in (2).

Proof. Differentiate $y(t_0 + h)$ in (5) with respect to h and set $h = 0$. □

Definition 2.2 (Lie affine algebra)**.** We define the Lie affine algebra of degree d as containing the elements $(\alpha, \beta^{[0]}, \ldots, \beta^{[d-1]}, \lambda) \in \mathrm{M}_n(\mathbf{R}) \times (\mathbf{R}^n)^d \times \mathbf{R}$. Addition and scalar multiplication are defined trivially elementwise. The bracket operation is

$$\Big[(\alpha_1, \beta_1^{[0]}, \ldots, \beta_1^{[d-1]}, \lambda_1), (\alpha_2, \beta_2^{[0]}, \ldots, \beta_2^{[d-1]}, \lambda_2)\Big] = ([\alpha_1, \alpha_2], \gamma^{[0]}, \ldots, \gamma^{[d-1]}, 0) \tag{6}$$

where

$$\gamma^{[j]} = \alpha_1 \beta_2^{[j]} - \alpha_2 \beta_1^{[j]} + \sum_{k=1}^{d-1-j} \left(\beta_1^{[j+k]} \frac{\lambda_2^k}{k!} - \beta_2^{[j+k]} \frac{\lambda_1^k}{k!} \right).$$

This algebra is a Lie algebra as both the skew-symmetry and the Jacobi identity can be proved. This is not surprising due to the connection with the vector field in (4) which will be presented in Lemma 2.5.

The scalar λ in the algebra is necessary for $d \geq 2$ to have a notion of unity time when this algebra is used to solve non-autonomous differential equations. Choosing $\lambda \neq 1$ and rescaling $\beta^{[j]}$'s accordingly, is equivalent to scaling the time parameter in (4), as will be demonstrated in Remark 2.6. For $d = 1$, the parameter λ is insignificant and could be removed from the definitions and expressions.

Definition 2.3 (Affine group)**.** The affine group of degree d is defined as containing the elements $(A, b^{[0]}, \ldots, b^{[d-1]}, \lambda) \in \mathrm{GL}_n(\mathbf{R}) \rtimes (\mathbf{R}^n)^d \times \mathbf{R}$.

The group product is

$$(A_2, b_2^{[0]}, \ldots, b_2^{[d-1]}, \lambda_2) \cdot (A_1, b_1^{[0]}, \ldots, b_1^{[d-1]}, \lambda_1) = (A_2 A_1, \xi^{[0]}, \ldots, \xi^{[d-1]}, \lambda_1 + \lambda_2) \tag{7}$$

where

$$\xi^{[j]} = A_2 b_1^{[j]} + \sum_{k=0}^{d-1-j} b_2^{[j+k]} \frac{\lambda_1^k}{k!}.$$

The group identity is written

$$\mathrm{Id} = (I, \mathbf{0}, \ldots, \mathbf{0}, 0).$$

Moreover the group inverse is

$$(A, b^{[0]}, \ldots, b^{[d-1]}, \lambda)^{-1} = (A^{-1}, \chi^{[0]}, \ldots, \chi^{[d-1]}, -\lambda) \tag{8}$$

where

$$\chi^{[j]} = -A^{-1} \sum_{k=0}^{d-1-j} b^{[j+k]} \frac{(-\lambda)^k}{k!}.$$

Elements of the affine group act on $\mathbf{R}^n \times \mathbf{R}$ as follows

$$(A, b^{[0]}, \ldots, b^{[d-1]}, \lambda) \cdot (y, t) = \left(Ay + \sum_{j=0}^{d-1} b^{[j]} \frac{t^j}{j!}, t + \lambda \right). \tag{9}$$

We define the following map from the affine algebra to the affine group,

$$\mathrm{Exp}(\alpha, \beta^{[0]}, \ldots, \beta^{[d-1]}, \lambda) = (\mathrm{e}^{\alpha}, b^{[0]}, \ldots, b^{[d-1]}, \lambda) \tag{10}$$

where

$$b^{[j]} = \sum_{k=0}^{d-1-j} \lambda^k \varphi_{k+1}(\alpha) \beta^{[k+j]}.$$

Lemma 2.5 establishes that this map really is the exponential as it represents the one-parameter subgroups in the affine group.

Lemma 2.4 (Logarithm map)**.**

$$\mathrm{Log}(A, b^{[0]}, \dots, b^{[d-1]}, \lambda) = (\log(A), \beta^{[0]}, \dots, \beta^{[d-1]}, \lambda)$$

where

$$\beta^{[j]} = \sum_{k=0}^{d-1-j} \lambda^k c_{k+1} b^{[k+j]}.$$

The coefficients c_k obey the recursive formula

$$c_{k+1} = -\varphi_1^{-1}(\log A) \sum_{i=0}^{k-1} \varphi_{k+1-i}(\log A) c_{i+1}, \qquad k \geq 1, \quad c_1 = \varphi_1^{-1}(\log A). \tag{11}$$

A must be sufficiently close to the identity $I \in \mathrm{GL}_n(\mathbf{R})$.

The first few c_k coefficients are

$$\begin{aligned}
c_1 &= \varphi_1^{-1} \\
c_2 &= -\varphi_1^{-2}\varphi_2 \\
c_3 &= \varphi_1^{-3}\varphi_2^2 - \varphi_1^{-2}\varphi_3 \\
c_4 &= -\varphi_1^{-2}\varphi_4 + 2\varphi_1^{-3}\varphi_2\varphi_3 - \varphi_1^{-4}\varphi_2^3 \\
c_5 &= -\varphi_1^{-2}\varphi_5 + \varphi_1^{-3}\varphi_2\varphi_4 - \varphi_1^{-4}\varphi_2^2\varphi_3 + \varphi_1^{-3}\varphi_3^2 + \varphi_1^{-5}\varphi_2\varphi_4 - 2\varphi_1^{-4}\varphi_2^2\varphi_3 + \varphi_1^{-2}\varphi_2^4
\end{aligned}$$

where all φ functions and their inverses are evaluated at $\log A$.

Proof. Assume d fixed. We will prove that the recursion (11) is necessary for $\mathrm{Exp}\circ\mathrm{Log} = \mathrm{Id}$ to hold. In this proof, all φ functions are evaluated at $\log A$.

$$\begin{aligned}
\mathrm{Exp} \circ \mathrm{Log}(A, b^{[0]}, \dots, b^{[d-1]}, \lambda) &= (A, b^{[0]}, \dots, b^{[d-1]}, \lambda) \\
\mathrm{Exp}(\log A, \beta^{[0]}, \dots, \beta^{[d-1]}, \lambda) &= (A, b^{[0]}, \dots, b^{[d-1]}, \lambda) \\
(A, \bar{b}^{[0]}, \dots, \bar{b}^{[d-1]}, \lambda) &= (A, b^{[0]}, \dots, b^{[d-1]}, \lambda)
\end{aligned}$$

which means that $\bar{b}^{[j]} = b^{[j]}$ must hold for $0 \leq j \leq d-1$. Furthermore

$$\begin{aligned}
\bar{b}^{[j]} &= \sum_{k=0}^{d-1-j} \lambda^k \varphi_{k+1} \beta^{[j+k]} \\
&= \sum_{k=0}^{d-1-j} \lambda^k \varphi_{k+1} \sum_{i=0}^{d-1-(j+k)} \lambda^i c_{i+1} b^{[i+j+k]} \\
&= \sum_{m=0}^{d-1-j} \lambda^m \left[\sum_{i=0}^{m} \varphi_{m-i+1} c_{i+1} \right] b^{[j+m]}
\end{aligned}$$

using $m = k + i$. Letting $j = d-1$, this immediately yields $c_1 = \varphi_1^{-1}$ if $\bar{b}^{[d-1]} = b^{[d-1]}$. Require $\bar{b}^{[d-1-k]} = b^{[d-1-k]}$ for $k = 1, 2, \dots, d-1$. In

$$\bar{b}^{[d-1-k]} = \sum_{m=0}^{k} \lambda^m \sum_{i=0}^{m} \varphi_{m-i+1} c_{i+1} b^{[d-1-k+m]}$$

the coefficient in front of λ^k must be zero. This means

$$0 = \sum_{i=0}^{k} \varphi_{k+1-i} c_{i+1} = \varphi_1 c_{k+1} + \sum_{i=0}^{k-1} \varphi_{k+1-i} c_{i+1}$$

and immediately gives the recursion (11). A similar calculation gives that (11) must also hold for $\mathrm{Log} \circ \mathrm{Exp} = \mathrm{Id}$. We have used that all φ functions and their inverses commute with each other. $\square$

The proof of Lemma 2.4 tacitly assumes that the affine algebra of degree $d-1$ is a subalgebra of the affine algebra of degree d, and similarly for the groups. The proof of this is omitted, but is easily seen in the matrix representation in Remark 2.10.

Lemma 2.5.

$$\mathrm{Exp}(h(\alpha, \beta^{[0]}, \dots, \beta^{[d-1]}, 1)) \cdot (y(t_0), t_0) = (y(t_0 + h), t_0 + h)$$

where $y(t)$ *satisfies* (4).

Proof. Using the exponential map (10) and the group action (9) we write

$$\begin{aligned} &\mathrm{Exp}(h(\alpha, \beta^{[0]}, \dots, \beta^{[d-1]}, 1)) \cdot (y(t_0), t_0) \\ &\qquad = \left(\mathrm{e}^{h\alpha} y(t_0) + \sum_{j=0}^{d-1} \sum_{k=0}^{d-1-j} h^k \varphi_{k+1}(h\alpha) h \beta^{[k+j]} \frac{t_0^j}{j!} \,,\; t_0 + h \right) \end{aligned}$$

Changing outer summation variable from j to $i = k + j$ we obtain

$$\left(\mathrm{e}^{h\alpha} y(t_0) + \sum_{i=0}^{d-1} \sum_{k=0}^{i} \frac{t_0^{i-k}}{(i-k)!} h^{k+1} \varphi_{k+1}(h\alpha) \beta^{[i]} \,,\; t_0 + h \right)$$

in which the first element corresponds to $y(t_0 + h)$ in (5). $\square$

Remark 2.6. If we scale time in (4) by $t = \gamma\tau$, we obtain the equation

$$\tilde{y}'(\tau) = \gamma\alpha y + \sum_{j=0}^{d-1} \frac{\tau^j}{j!} \gamma^{j+1} \beta^{[j]}, \qquad \tilde{y}(\tau_0) = y(t_0) = y_0$$

Using this and Lemma 2.5, one sees that

$$\mathrm{Exp}(h(\alpha, \beta^{[0]}, \gamma\beta^{[1]}, \dots, \gamma^{d-1}\beta^{[d-1]}, 1/\gamma)) \cdot (\tilde{y}(\tau_0), \tau_0) = (y(t_0 + h), t_0 + h)$$

thus illustrating a one-parameter isotropy subalgebra [1, 9] parametrized by γ.

Lemma 2.7.

$$\varphi_\ell((\gamma_1 + \gamma_2)\alpha) = \frac{1}{(\gamma_1 + \gamma_2)^\ell} \left(\mathrm{e}^{\gamma_1 \alpha} \gamma_2^\ell \varphi_\ell(\gamma_2 \alpha) + \sum_{k=1}^{\ell} \frac{\gamma_1^{\ell-k} \gamma_2^k}{(\ell-k)!} \varphi_k(\gamma_2 \alpha) \right)$$

where γ_1 *and* γ_2 *are real numbers.*

Proof. Let $\bar{\gamma} = \gamma_1 + \gamma_2$. By the homomorphism property of the exponential, we have

$$\mathrm{Exp}(\bar{\gamma}(\alpha, \beta^{[0]}, \dots, \beta^{[d-1]}, \lambda)) = \\ \mathrm{Exp}(\gamma_1(\alpha, \beta^{[0]}, \dots, \beta^{[d-1]}, \lambda)) \cdot \mathrm{Exp}(\gamma_2(\alpha, \beta^{[0]}, \dots, \beta^{[d-1]}, \lambda)) \quad (12)$$

The left hand side of (12) is the group element $\left(\mathrm{e}^{\bar{\gamma}\alpha}, b_L^{[0]}, \dots, b_L^{[d-1]}, \bar{\gamma}\lambda\right)$ where

$$b_L^{[j]} = \sum_{k=0}^{d-1-j} (\bar{\gamma}\lambda)^k \varphi_{k+1}(\bar{\gamma}\alpha)\bar{\gamma}\beta^{[k+j]}$$

and the right hand side of (12) will be $(\mathrm{e}^{\bar{\gamma}\alpha}, b_R^{[1]}, \dots, b_R^{[d-1]}, \bar{\gamma}\lambda)$ where

$$b_R^{[j]} = \mathrm{e}^{\gamma_1\alpha}\left(\sum_{k=0}^{d-1-j} (\gamma_2\lambda)^k \varphi_{k+1}(\gamma_2\alpha)\gamma_2\beta^{[k+j]}\right) \\ + \sum_{k=0}^{d-1-j} \frac{(\gamma_1\lambda)^k}{k!} \sum_{i=0}^{d-1-(j+k)} (\gamma_2\lambda)^i \varphi_{i+1}(\gamma_2\alpha)\gamma_2\beta^{[i+j+k]}$$

in which the double sum is rearranged using $l = i + k$ to obtain

$$b_R^{[j]} = \mathrm{e}^{\gamma_1\alpha}\left(\sum_{k=0}^{d-1-j} (\gamma_2\lambda)^k \varphi_{k+1}(\gamma_2\alpha)\gamma_2\beta^{[k+j]}\right) \\ + \sum_{l=0}^{d-1} \frac{(\gamma_1\lambda)^{(l-i)}}{(l-i)!} \sum_{i=0}^{d-1-l} (\gamma_2\lambda)^i \varphi_{i+1}(\gamma_2\alpha)\gamma_2\beta^{[i+l]}.$$

The relation for φ_ℓ, $1 \le \ell \le d$ is now obtained by comparing the coefficients of λ^ℓ in $b_L^{[0]} = b_R^{[0]}$. □

The φ functions possess a double-angle relation. This relation is a building block of the scaling and corrected squaring approach of calculating the φ functions numerically.

Corollary 2.8 (Squaring of φ functions)**.** The φ functions (2) have the following squaring property,

$$\varphi_\ell(2\alpha) = \frac{1}{2^\ell}\left(\mathrm{e}^\alpha \varphi_\ell(\alpha) + \sum_{k=1}^{\ell} \frac{1}{(\ell-k)!}\varphi_k(\alpha)\right)$$

Proof. Set $\gamma_1 = \gamma_2 = 1$ in Lemma 2.7. □

This squaring property is also attainable by dividing the integration interval in (2) in half and then shifting the integration variable in the second half, as outlined in [6].

The following corollary will be applied in further studies on φ functions.

Corollary 2.9.

$$\mathrm{e}^{-\alpha}\varphi_\ell(\alpha) = \sum_{k=1}^{\ell} \frac{(-1)^{k+1}}{(\ell-k)!}\varphi_k(-\alpha)$$

Proof. Set $\gamma_1 = -1$ and $\gamma_2 = 1$ in Lemma 2.7. □

Remark 2.10. According to Ado's theorem, we have that the Lie affine algebra and affine group are isomorphic to a subalgebra of matrices in $\mathrm{M}_{n+d}(\mathbf{R})$ and to a subgroup of $\mathrm{GL}_{n+d}(\mathbf{R})$. The isomorphism for the Lie affine algebra can be represented by the map

$$(\alpha, \beta^{[0]}, \dots, \beta^{[d-1]}, \lambda) \mapsto \begin{pmatrix} \alpha & \beta^{[0]} & \beta^{[1]} & \cdots & \beta^{[d-1]} \\ \mathbf{0}^T & 0 & 0 & \cdots & 0 \\ \mathbf{0}^T & \lambda & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0}^T & 0 & \cdots & \lambda & 0 \end{pmatrix} \tag{13}$$

where $\mathbf{0}$ is a column vector of size n.

The isomorphism for the affine group can be represented by the map

$$(A, b^{[0]}, \dots, b^{[d-1]}, \lambda) \mapsto \begin{pmatrix} A & b^{[0]} & b^{[1]} & \dots & & b^{[d-1]} \\ \mathbf{0}^T & 1 & 0 & \dots & & 0 \\ \mathbf{0}^T & \lambda & \ddots & & & \\ \vdots & & \ddots & \ddots & & \\ \mathbf{0}^T & \frac{\lambda^{d-1}}{(d-1)!} & & \ddots & \ddots & 0 \\ \mathbf{0}^T & \frac{\lambda^d}{d!} & \frac{\lambda^{d-1}}{(d-1)!} & \dots & \lambda & 1 \end{pmatrix} \tag{14}$$

where elements below the diagonal are

$$c_{i,j} = \frac{\lambda^{i-j}}{(i-j)!} \qquad \text{for } 2 \leq j \leq d+1 \text{ and } i \geq j.$$

3. Affine Lie group integrators

In this section, we extend the Lie group integrators presented in [11] using a second degree affine group on a scalar equation. This is an exemplification of the results presented in Section 2.

Let a non-autonomous differential equation be given by

$$y'(t) = \alpha y + \beta^{[0]} + t\beta^{[1]} \qquad y(t_0) = y_0 \in \mathbf{R}^n, \quad \alpha \in \mathbf{R}^{n \times n}, \quad \beta^{[j]} \in \mathbf{R}^n \tag{15}$$

In order to construct a Runge–Kutta–Munthe-Kaas (RKMK) scheme, one needs to define a configuration space (a manifold M) for the solution, a map from the configuration space to an algebra $\mathfrak{g}$, and an action from the algebra on the configuration space. Our problem is non-autonomous, so the time parameter t must be included in the configuration space, we write $M = \mathbf{R}^n \times \mathbf{R}$. The map $f\colon M \to \mathfrak{g}$ can be chosen to be

$$f\colon (y, t) \mapsto (\alpha, \beta^{[0]}, \beta^{[1]}, 1) \tag{16}$$

Alternative maps using the isotropy subalgebra are given in Remark 2.6.

In RKMK schemes, calculations (inner stages) are performed in the algebra $\mathfrak{g}$ and the result is subsequently obtained via the algebra action which factors into the exponential map and the group action. If we want to implement Lie–Euler for the problem (15) using (16), we would get the scheme

$$(y_{n+1}, t_{n+1}) = \mathrm{Exp}(h(\alpha, \beta^{[0]}, \beta^{[1]}, 1)) \cdot (y_n, t_n) \tag{17}$$

and by inserting explicit expressions from (10) and (9) (or from Section A.1), one obtains

$$(y_{n+1}, t_{n+1}) = \Big(\mathrm{e}^{h\alpha} y_n + h\varphi_1(h\alpha)\beta^{[0]} + h\varphi_1(h\alpha)\beta^{[1]} + h^2\varphi_2(h\alpha)\beta^{[1]}, t_n + h\Big) \tag{18}$$

which can be coined Lie–Euler of degree 2.

The report [10] constructs a fourth order commutator free scheme using a second degree ($d = 2$) affine group for a non-autonomous problem. The approach there to find $f\colon M \to \mathfrak{g}$ is to linearize the nonlinear function,

$$N(y,t) \approx N_n + t\frac{N(y,t) - N_n}{t} = N^{[0]} + tN^{[1]}$$

where $N_n = N(y_n, t_n)$. Given this, a commutator free scheme of degree 2 is constructed using the exponential map and group action that corresponds to the group chosen. This scheme is compared numerically with the fourth order commutator free scheme obtained by using only $N^{[0]}$ above and the affine group of degree $d = 1$. The numerical results therein indicate that the commutator free scheme using $d = 2$ has a slight advantage over the scheme with $d = 1$.

For arbitrary nonlinear function, a Taylor expansion in t is needed to be able to apply arbitrary degree affine Lie group integrators. This may or may not be feasible depending on the way $N(y,t)$ is presented. For this reason, it is not believed that arbitrary degree affine Lie group integrators can contribute much to efficient numerical solution of differential equations.

4. Discussion

Generalized affine groups of arbitrary degree d have been constructed and the role of higher degree φ functions have been exemplified. It is not believed that Lie group integrators using high degree affine groups can benefit substantially with regard to numerical performance compared to using the first degree group, given a general $N(y,t)$ in (4), but it is hoped that this generalization can provide a tool for further analysis on exponential integrators.

The intention of this study was rather to reveal a structure in which the φ functions appear and to possibly be able to use this structure in future studies of φ functions. A natural step forward is to see if backward error analysis of the φ functions is possible analogously to work of N. Higham in [5] on backward error analysis of Padé approximants to the exponential function, in which precisely the group and algebra structure of general linear groups are used. This is the task of the forthcoming paper [4].

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A. Explicit low-degree groups

This appendix lists the definitions and operations specified in Section 2 explicitly when the group degree d is fixed.

A.1. First degree, $d = 1$

The parameter λ plays no role for $d = 1$ and is thus superflous. It has been kept in these expression in order to match precisely with the general results in Section 2.

Group:	$\mathrm{GL}_n(\mathbf{R}) \rtimes \mathbf{R}^n \times \mathbf{R}$
Group identity:	$(I, \mathbf{0}, 0)$
Group product:	$(A_2, b_2, \lambda_2) \cdot (A_1, b_1, \lambda_1) = (A_2A_1, A_2b_1 + b_2, \lambda_1 + \lambda_2)$
Group inverse:	$(A, b, \lambda)^{-1} = (A^{-1}, -A^{-1}b, -\lambda)$
Algebra:	$\mathrm{M}_n \times \mathbf{R}^n \times \mathbf{R}$
Algebra addition:	$(\alpha_1, \beta_1, \lambda_1) + (\alpha_2, \beta_2, \lambda_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \lambda_1 + \lambda_2)$
Algebra bracket:	$[(\alpha_1, \beta_1, \lambda_1), (\alpha_2, \beta_2, \lambda_2)] = ([\alpha_1, \alpha_2], \alpha_1\beta_2 - \alpha_2\beta_1, 0)$
Exponential map:	$\mathrm{Exp}(\alpha, \beta, \lambda) = (\mathrm{e}^{\alpha}, \phi_1(\alpha)\beta, \lambda)$
Logarithm:	$\mathrm{Log}(A, b, \lambda) = \left(\log A, \frac{\log A}{A-I}b, \lambda\right) = (\log A, \varphi_1^{-1}(\log A)b, \lambda)$
Group action:	$(A, b, \lambda) \cdot (y, t) = (Ay + b, t + \lambda)$

A.2. Second degree, $d = 2$

Group:	$\mathrm{GL}_n(\mathbf{R}) \rtimes (\mathbf{R}^n \times \mathbf{R}^n) \times \mathbf{R}$
Group identity:	$(I, \mathbf{0}, \mathbf{0}, 0)$
Group product:	$(A_2, b_2^{[0]}, b_2^{[1]}, \lambda_2) \cdot (A_1, b_1^{[0]}, b_1^{[1]}, \lambda_1)$ $= \left(A_2A_1, A_2b_1^{[0]} + b_2^{[0]}, A_2b_1^{[1]} + b_2^{[1]}, \lambda_1 + \lambda_2\right)$
Group inverse:	$(A, b^{[0]}, b^{[1]})^{-1} = \left(A^{-1}, -A^{-1}(b^{[0]} - b^{[1]}\lambda), -A^{-1}b^{[1]}, -\lambda\right)$
Algebra:	$\mathrm{M}_n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$
Algebra addition:	$(\alpha_1, \beta_1^{[0]}, \beta_1^{[1]}, \lambda_1) + (\alpha_2, \beta_2^{[0]}, \beta_2^{[1]}, \lambda_2)$ $= \left(\alpha_1 + \alpha_2, \beta_1^{[0]} + \beta_2^{[0]}, \beta_1^{[1]} + \beta_2^{[1]} \lambda_1 + \lambda_2\right)$
Algebra bracket:	$[(\alpha_1, \beta_1^{[0]}, \beta_1^{[1]}, \lambda_1), (\alpha_2, \beta_2^{[0]}, \beta_2^{[1]}, \lambda_2)]$ $= ([\alpha_1, \alpha_2], \alpha_1\beta_2^{[0]} - \alpha_2\beta_1^{[0]} + \beta_1^{[1]}\lambda_2 - \beta_2^{[1]}\lambda_1, \alpha_1\beta_2^{[1]} - \alpha_2\beta_1^{[1]}, 0)$
Exponential map:	$\mathrm{Exp}(\alpha, \beta^{[0]}, \beta^{[1]}, \lambda)$ $= (\mathrm{e}^{\alpha}, \phi_1(\alpha)\beta^{[0]} + \lambda\phi_2(\alpha)\beta^{[1]}, \phi_1(\alpha)\beta^{[1]}, \lambda)$
Logarithm:	$\mathrm{Log}(A, b^{[0]}, b^{[1]}, \lambda)$ $= (\log(A), \frac{(A\log A - \log A)b^{[0]} + (\log A - A + 1)b^{[1]}\lambda)}{(A-I)^2}, \frac{\log A}{A-I}b^{[1]}, \lambda)$ $= (\log A, \varphi_1^{-1}(\log A)b^{[0]} - \varphi_1^{-2}(\log A)\varphi_2(\log A)b^{[1]}\lambda,$ $\varphi_1^{-1}(\log A)b^{[1]}, \lambda)$
Group action:	$(A, b^{[0]}, b^{[1]}, \lambda) \cdot (y, t) = (Ay + b^{[0]} + tb^{[1]}, t + \lambda)$

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