The Trend-Renewal Process for Statistical Analysis of Repairable Systems

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Abstract

A repairable system can briefly be characterized as a system which is repaired rather than replaced after a failure. The most commonly used models for the failure process of a repairable system are nonhomogeneous Poisson processes (NHPP), corresponding to minimal repairs, and renewal processes (RP), corresponding to perfect repairs. This paper introduces and studies a more general model called the Trend-Renewal Process (TRP). The TRP is a time transformed renewal process, having both the ordinary renewal process (RP) and the nonhomogeneous Poisson process (NHPP) as special cases. Parametric inference in the TRP model is studied both for observation of single systems and of several systems, with a possible unobserved heterogeneity between the systems.

1 Introduction

Lawless & Thiagarajah (1996) discussed statistical inference for recurrent events which incorporated both time trends and renewal type behaviour. Such models are of interest for example in repairable systems reliability as a supplement to the widely used nonhomogeneous Poisson process (NHPP) and renewal process (RP) models, which strictly are applicable only under minimal repair and perfect repair conditions, respectively.

In this paper we study a related class of models, called the trend-renewal process (TRP), which can be defined as time-transformed renewal processes and therefore clearly include both the NHPP and RP models as special cases. A particular example of a trend-renewal process is the *inhomogeneous gamma process* of Berman (1981), but in general the TRP allows a general positive distribution where Berman used a gamma distribution. A closely related model is the modulated renewal process of Cox (1972). It turns out, however, that the TRP is not included in, nor does include, the modulated renewal process. This also holds for the TRP in comparison to the general models of Berman & Turner (1992) and Lawless & Thiagarajah (1996).

The TRP as defined here was first described in Lindqvist (1993) and was used as an alternative to the NHPP in a simulation study of robustness of trend tests for repairable systems (Lindqvist, Kjønstad & Meland, 1994). In fact, it follows from its definition that the TRP is particularly easy to simulate if reasonable alternatives to an NHPP or RP are wanted.

In the present paper the main emphasis is on statistical inference in parametric TRP models. In particular we shall consider inference in cases where several independent sequences of events are observed. This is of interest in applications involving the observation of recurrent events for several individuals at the same time, e.g. machines of the same type. Models for such situations may involve individual effects, either observed or non-observed ones. For example, such effects could be caused by the installation of machines in different environments. Lawless (1987) and Follmann & Goldberg (1988) consider, respectively, NHPP and RP models for such situations. In the present paper we suggest corresponding models for the TRP process in the case of non-observed heterogeneity.

The motivation for the present study has been modelling and analysis of failure data for repairable systems. Thus, although most of the models and methods studied will have wider applicability to phenomena involving recurrent events, we shall find it convenient to use the reliability terminology in the following. For further information on repairable systems reliability we refer to the book by Ascher & Feingold (1984).

As an alternative to the models mentioned above, Ascher & Feingold (1984) review models for which the times between events are independent but not identically distributed. A drawback of these models turns out to be the assumed independence of interfailure times. As argued for example in Berman (1981) and Thompson (1988), successive intervals between failures of a repairable system will usually be stochastically dependent under aging. Note that the TRP, and even the NHPP, has the property of dependent interfailure times.

The literature on reliability also contains a class of models often called "imperfect repair" models, the classical one being the model of Brown & Proschan (1983), which assumes that at the time of each failure a perfect repair occurs with probability p and a minimal repair occurs with probability $1 - p$. In a recent paper, Dorado, Hollander & Sethuraman (1997) suggest a general repair model which contains several known models of this kind as special cases. The statistical analyses of these models, however, assume that information on the repair action is available at each failure in addition to the failure times, while the analyses presented in this paper use information only on the observed failure times.

2 Repairable Systems: Notation and Definitions

Recall that a repairable system is defined to be a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method, other than replacement of the entire system (Ascher & Feingold, 1984).

Consider a repairable system, observed from time $t = 0$, say. Denote by $T_1, T_2, \ldots$ the successive *failure times*, and let the times *between* failures be denoted $X_1, X_2, \ldots$. Thus we have

$$X_i = T_i - T_{i-1}, \quad i = 1, 2, \ldots$$

where for convenience we define $T_0 \equiv 0$. The sequence $T_1, T_2, \dots$ of failure times and the sequence $X_1, X_2, \dots$ of interfailure times thus contain exactly the same information about a particular realization of the process. This also holds for the counting process representation

$$N(t) = \text{ number of failures in } (0, t]$$

defined for all $t > 0$.

Above we have tacitly assumed that all repair times equal 0. This is a reasonable assumption if repair times are negligible compared to times between failures, or if the time scale is taken to be the *operation time* of the system. Note that in the latter case, a marked point process may keep track of the repair times, and possibly additional information such as production conditions, maintenance and external conditions. However, marked processes will not be considered in this paper.

In order to mathematically describe models for general repairable systems, we introduce some more notation from the theory of point processes. Let $\mathcal{F}_{t-}$ denote the *history* of the failure process up to, but not including, t. Usually we shall think of $\mathcal{F}_{t-}$ as generated by $\{N(s) : 0 \leq s < t\}$ thus keeping record of all the failures before time t. The $\mathcal{F}_{t-}$ could, however, contain more information, for example in a marked point process. The conditional intensity $\lambda(t|\mathcal{F}_{t-})$ is defined (Andersen, Borgan, Gill & Keiding, 1993) by

$$\lambda(t|\mathcal{F}_{t-}) = \lim_{\Delta t \downarrow 0} \frac{Pr(\text{failure in } [t, t+\Delta t)|\mathcal{F}_{t-})}{\Delta t} \; ; \; t > 0.$$

Thus, $\lambda(t|\mathcal{F}_{t-})\Delta t$ is approximately the probability of failure in the time interval $[t, t+\Delta t)$, conditional on the failure history before time t. Note that the so called ROCOF (rate of occurrence of failures, (Ascher & Feingold, 1984)) is the corresponding unconditional intensity.

For a renewal process with interarrival distribution F, RP(F), it is well known that

$$\lambda(t|\mathcal{F}_{t-}) = z(t - T_{N(t-)})$$

where z is the hazard rate corresponding to F, and $t - T_{N(t-)}$ is the time since the last failure strictly before time t. Thus, after each failure, the conditional rate of failures is returned to the state at time 0, i.e. the system is as good as new immediately after each failure.

On the other hand, an NHPP with intensity function $\lambda(t)$ (for short, NHPP($\lambda(\cdot)$)) has the conditional intensity

$$\lambda(t|\mathcal{F}_{t-}) = \lambda(t)$$

Here we can again think of $\lambda(\cdot)$ as the hazard rate of the time to *first* failure, but now the conditional rate of failures after a failure is exactly as if no failure had ever occurred. Thus the system, after a failure, is only restored to a state where it is exactly as good (or bad) as it was immediately before the failure.

In this paper we shall study processes, either single, or several similar processes observed simultaneously, each observed on (possibly different) time intervals of the form $[0, \tau]$, where the τ are stopping times, and the observed failure times for a specific system are denoted $T_1, T_2, \ldots, T_{N(\tau)}$. The histories $\mathcal{F}_{t-}$ are in the following always assumed to be generated by $\{N(s) : 0 \leq s < t\}$.

A general expression of the likelihood function for given observations $T_1, T_2, \ldots, T_{N(\tau)}$ is then (Andersen *et al.*, 1993),

$$L = \{\prod_{i=1}^{N(\tau)} \lambda(T_i|\mathcal{F}_{T_i-})\} \exp(-\int_0^\tau \lambda(u|\mathcal{F}_{u-})du) \tag{1}$$

For the RP and NHPP processes, (1) simplifies to the well known likelihoods for these cases (see e.g. Crowder, Kimber, Smith & Sweeting (1991)).

3 The Trend-Renewal Process

The TRP process generalizes the following well known property of the NHPP:
If $T_1, T_2, \ldots$ is an NHPP($\lambda(\cdot)$), then the time-transformed stochastic process $\Lambda(T_1), \Lambda(T_2), \ldots$ is HPP(1). Here $\Lambda(t) = \int_0^t \lambda(u)du$ is the cumulative intensity, and HPP(λ) denotes a homogeneous Poisson process with constant intensity λ.

We define the *trend-renewal process* (TRP) simply by allowing the above HPP(1) to be any renewal process RP(F) where the distribution F has expected value equal to 1. Thus, in addition to the intensity function $\lambda(t)$, which is all we need to specify an NHPP, we need to specify a distribution function F. Formally we can define the process TRP($F, \lambda(\cdot)$) as follows.

Definition 1 *Let $\lambda(t)$ be a nonnegative function defined for $t \geq 0$, with $\Lambda(t) = \int_0^t \lambda(u)du$, and let F be a positive distribution with expected value 1. The process $T_1, T_2, \ldots$ is called TRP($F, \lambda(\cdot)$) if the time-transformed process $\Lambda(T_1), \Lambda(T_2), \ldots$ is RP(F), i.e. if the $\Lambda(T_i) - \Lambda(T_{i-1})$; $i = 1, 2, \ldots$ are i.i.d. with distribution function F.*

Figure 1 illustrates the definition. Note that for an NHPP($\lambda(\cdot)$), the RP(F) is replaced by HPP(1). Thus TRP($\exp, \lambda(\cdot)$) = NHPP($\lambda(\cdot)$), which shows that the TRP generalizes the NHPP. (Here 'exp' means the unit exponential distribution.)

The requirement that F has expected value 1 is made for convenience. The scale is then taken care of by the intensity function $\lambda(t)$.

For consistency of notation, we shall in the following find it convenient to represent renewal processes in general by RP(F, λ), where F has expected value 1 and the constant λ equals the inverse expected value of the interarrival distribution. The distribution function of the interarrival distribution is thus $G(t) = F(\lambda t)$. With this convention we have TRP(F, λ) = RP(F, λ) when λ is a constant.

We obtain the conditional intensity function for the TRP($F, \lambda(\cdot)$) from the following derivation, where $z(t)$ denotes the hazard rate corresponding to F.

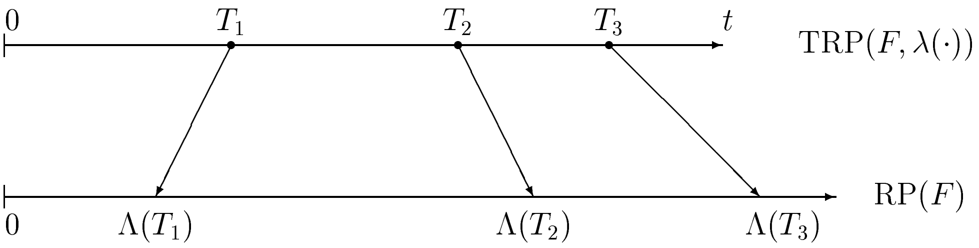

Figure 1: The defining property of the trend-renewal process.

$$\begin{aligned}
\lambda(t|\mathcal{F}_{t-})\Delta t &\doteq P(\text{failure in } [t, t+\Delta t)|\mathcal{F}_{t-}) \\
&= P(\text{failure in } RP(F) \text{ in } [\Lambda(t), \Lambda(t+\Delta t))|\mathcal{F}_{t-}) \\
&\doteq P(\text{failure in } RP(F) \text{ in } [\Lambda(t), \Lambda(t)+\lambda(t)\Delta t))|\mathcal{F}_{t-}) \\
&= z(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t)\Delta(t)
\end{aligned}$$

so that

$$\lambda(t|\mathcal{F}_{t-}) = z(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t) \tag{2}$$

It is thus seen that the conditional intensity function of the TRP is a product of one factor, $\lambda(t)$, which depends on the age t of the system and one factor which depends on the (transformed) time from the last previous failure. In fact, in (2) time since last failure is measured on a scale depending on the cumulative aging effect. This is in contrast to Cox's modulated renewal process (Cox, 1972) and the model of Lawless & Thiagarajah (1996), which instead use conditional intensities of the form $z(t - T_{N(t-)})\lambda(t)$.

As a motivation for the TRP model, suppose that a major part of the system is replaced at failures (for example the engine of a tractor, see section 8.3), while the rest of the system and thus the environment where the major part works, is not maintained. Then due to aging effects it is plausible that an ideal renewal process induced by replacements is transformed to an observed failure process $T_1, T_2, \ldots$ which is no longer a renewal process. In the TRP this is modelled by a "clock" which accelerates as time passes when $\lambda(t)$ is increasing in t.

More specifically, consider the formula (2) in the case when $z(t)$ as well as $\lambda(t)$ are increasing functions of t. The conditional failure intensity at a time s_0 *after* a failure at t_0 is by (2) equal to

$$z(\Lambda(t_0 + s_0) - \Lambda(t_0))\lambda(t_0 + s_0) \tag{3}$$

By the monotonicity assumptions both the z-factor and the λ-factor of (3) are increasing in t_0 for fixed s_0. For example, putting $s_0 = 0$ we conclude from (3) that a failure and repair at time t_0 results in the reduction in conditional intensity down to $z(0)\lambda(t_0)$, which increases with time if $z(0) > 0$. Thus aging *reduces* the short-time effect of repairs in this case. Moreover, (3) implies more generally that the effect of the repair or replacement, when considered at a fixed additional time s_0 after failure, is also decreasing with the age of the system (t_0).

3.1 Limit Results for TRP

Let as before $N(t)$ denote the number of failures in $(0, \tau]$ and define also $V(t) = E[N(t)]$. Assume further that $\Lambda(t)$ is monotonically increasing towards ∞ as $t \to \infty$.

It follows from the definition of TRP$(F, \lambda(\cdot))$ that $N^*(u) \equiv N(\Lambda^{-1}(u))$ $(u > 0)$ is a renewal process, RP$(F, 1)$. Moreover, if we define $V^*(u) = E[N^*(u)]$ then $V^*(u) = V(\Lambda^{-1}(u))$.

Standard asymptotic results for renewal processes (Ross, 1983) imply that $N^*(u)/u$ tends to 1 almost surely, and that $V^*(u)/u \to 1$ as $u \to \infty$. Now substituting $t = \Lambda^{-1}(u)$ gives the following limit results for the TRP,

$$\frac{N(t)}{\Lambda(t)} \to 1 \text{ a.s. }, \frac{V(t)}{\Lambda(t)} \to 1 \text{ as } t \to \infty$$

Thus we may, at least asymptotically, think of $\Lambda(t)$ as the expected number of failures until time t.

4 Statistical Inference in the TRP Model: Likelihood Function

4.1 Likelihood for a Single TRP

Suppose first that a single system has been observed in $[0, \tau]$, for a stopping time τ, with failures at $T_1, T_2, \ldots, T_{N(\tau)}$. If a TRP$(F, \lambda(\cdot))$ is used as a model, then substitution of (2) into (1) gives the likelihood

$$L = \{\prod_{i=1}^{N(\tau)} z(\Lambda(T_i) - \Lambda(T_{i-1}))\lambda(T_i)\} \exp\{-\int_0^\tau z(\Lambda(u) - \Lambda(T_{N(u-)})\lambda(u)du\} \tag{4}$$

where $z(\cdot)$ is the hazard rate corresponding to F.

Equivalently, if f is the density function corresponding to F, we can write the likelihood as

$$L = \{\prod_{i=1}^{N(\tau)} f(\Lambda(T_i) - \Lambda(T_{i-1}))\lambda(T_i)\}\{1 - F(\Lambda(\tau) - \Lambda(T_{N(\tau)}))\} \tag{5}$$

The latter form of the likelihood follows since the conditional density of T_i given $T_1 = t_1, \ldots, T_{i-1} = t_{i-1}$ is $f(\Lambda(t_i) - \Lambda(t_{i-1}))\lambda(t_i)$, and the probability of no failures in the time interval $(N(\tau), \tau]$, given $T_1, \ldots, T_{N(\tau)}$, is $1 - F(\Lambda(\tau) - \Lambda(T_{N(\tau)}))$.

4.2 Several Systems

Suppose that m systems are observed, where the jth system ($j = 1, 2, \ldots, m$) is observed in the time interval $[0, \tau_j]$. For the jth system, let N_j denote the number of failures that occur during the observation period, and let the specific failure times and interfailure times be denoted, respectively,

$$T_{1j} < T_{2j} < \cdots < T_{N_j j}$$

and

$$X_{1j}, X_{2j}, \ldots, X_{N_j j}$$

Even if the systems are considered to be of the same type, they may well exhibit different probability mechanisms. For example, systems may be used under varying environmental or operational conditions. If the model $\text{TRP}(F_j, \lambda_j(\cdot))$ is assumed for the jth system, then a likelihood L_j as in (4-5) is defined (with indices j on the F and $\lambda(\cdot)$). Moreover, if the systems operate independently of each other (an assumption we shall make throughout this paper), a full likelihood L is obtained as $L = \prod_{j=1}^{m} L_j$.

Note that if we restrict attention to systems modelled by NHPPs, then the F_j are uniquely given by $F(x) = 1 - \exp(-x)$. Thus in this case differences between systems are modelled entirely by letting the intensity functions $\lambda_j(t)$ vary from system to system. Motivated by this, for joint TRP modelling of several systems of the same kind we shall always assume that the system processes share a common F with expected value 1. Thus possible differences between systems will be modelled by varying the function $\lambda(t)$. More specifically, in the most general case we assume that the failures of the jth system follow the process $\text{TRP}(F, \lambda_j(\cdot))$, $j = 1, \ldots, m$.

The general form of the full likelihood is thus $L \equiv \prod_{j=1}^{m} L_j$ where

$$L_j = \{\prod_{i=1}^{N_j} f(\Lambda_j(T_{ij}) - \Lambda_j(T_{i-1,j}))\lambda_j(T_{ij})\}\{1 - F(\Lambda_j(\tau_j) - \Lambda_j(T_{N_j j}))\} \tag{6}$$

As an example of the use of (6), assume that differences between system performances can be attributed to an *observable* covariate vector $\mathbf{x}$, and that the intensity $\lambda_j(t)$ for system j is represented by a proportional hazards model with

$$\lambda_j(t) = g(\mathbf{x}_j)\lambda(t) \quad (j = 1, \ldots, m) \tag{7}$$

Here $\lambda(\cdot)$ is a basic intensity function common to all systems, while g is a function of the covariate vector $\mathbf{x}_j$ of system j. The special cases of NHPP and RP with covariates are studied, respectively, by Lawless (1987), and Follmann & Goldberg (1988).

Models with observable covariates will not be pursued in this paper. Instead we shall treat the case where there is possibly an *unobserved heterogeneity* between the systems (in the literature also called "individual effects" or "frailty"). Assuming that there are no observed covariates, it is common to modify (7) to

$$\lambda_j(t) = a_j\lambda(t) \tag{8}$$

where the a_j are *unobservable* random variables taking values independently across systems, according to a common probability distribution with distribution function H, say (Andersen *et al.* (1993), Chapter IX). For convenience we shall throughout this paper assume that the expected value of a_j equals 1. Thus in (8), $\lambda(\cdot)$ is the basic intensity function, while the a_j represent a possibly different intensity "level" for each system, averaging to 1. The special case when $a_j = 1$ with probability 1 will be referred to as the "no heterogeneity" case.

For given values of the a_j the likelihood for the jth system is now

$$L_j(a_j) = \{\prod_{i=1}^{N_j} f(a_j(\Lambda(T_{ij}) - \Lambda(T_{i-1,j})))a_j\lambda(T_{ij})\}\{1 - F(a_j(\Lambda(\tau_j) - \Lambda(T_{N_j j})))\} \tag{9}$$

and the full likelihood is, since the a_j are unobservable,

$$L = \prod_{j=1}^{m} \int L_j(a_j)dH(a_j) \tag{10}$$

We shall use the notation HTRP$(F, \lambda(\cdot), H)$ for the model with likelihood (10). Here F, H are distributions corresponding to positive random variables with expected value 1, while $\lambda(t)$ is a positive function defined for $t \geq 0$. We remark that the convention of expected values equal to 1 is done for uniqueness reasons, and is also convenient for the comparison of estimates of λ-functions under different models.

In practical applications it is often reasonable to include both observed and non-observed heterogeneity in the models, thus combining (7) and (8). As already noted, however, in this paper confine attention to the case of no covariates so that the HTRP described above is our most general model.

An important feature of the HTRP model is that several important models are easily represented in terms of the HTRP formulation as follows. Note that the notation HPP, NHPP, RP and TRP are used as before, and that we define corresponding models with heterogeneity as in (8) by putting an H in front of the abbreviations.

Specifically, from a full model, HTRP$(F, \lambda(\cdot), H)$, we identify the following seven submodels,

$$\text{HPP}(\lambda) = \text{HTRP}(\exp, \lambda, 1)$$

$$\text{RP}(F, \lambda) = \text{HTRP}(F, \lambda, 1)$$

$$\text{NHPP}(\lambda(\cdot)) = \text{HTRP}(\exp, \lambda(\cdot), 1)$$

$$\text{TRP}(F, \lambda(\cdot)) = \text{HTRP}(F, \lambda(\cdot), 1)$$

$$\text{HHPP}(\lambda, H) = \text{HTRP}(\exp, \lambda, H)$$

$$\text{HRP}(F, \lambda, H) = \text{HTRP}(F, \lambda, H)$$

$$\text{HNHPP}(\lambda(\cdot), H) = \text{HTRP}(\exp, \lambda(\cdot), H)$$

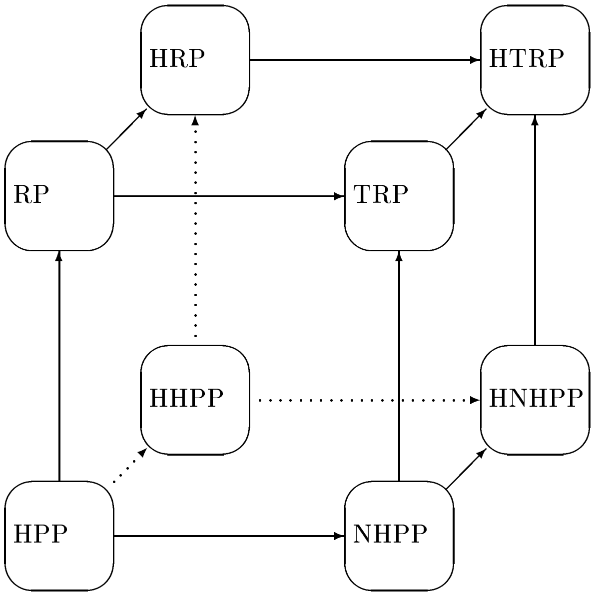

Figure 2: The model cube.

Here 'exp' means the exponential distribution with expectation 1 and 1 means the distribution degenerate at 1. Note also the distinction between the constant λ and the function $\lambda(t)$.

The eight models are depicted in Figure 2, where each vertex represents a model, and the lines connecting them correspond to changing one of the "coordinates" in the HTRP-notation. It is seen that going to the right corresponds to introducing a time trend, going up corresponds to going to a non-Poisson case and finally that going backwards (inwards) corresponds to introducing heterogeneity. In later examples we shall use this cube to give an easy presentation of maximum log likelihood values for the different models.

5 Parametric Modelling of HTRP

A general parametric model for an HTRP is obtained through defining parametric functions $F(y;\theta), \lambda(t;\theta)$, $H(a;\theta)$ in the likelihood (10). Here $\theta \in \Theta$ is usually a vector which can be decomposed as $\theta = (\theta_F, \theta_\lambda, \theta_H)$ where F depends on θ only through θ_F etc. Moreover, the parameterization of the HTRP is assumed to be such that the each vertex of the likelihood cube (Figure 2) corresponds to a subset of Θ, usually obtained by fixing some components of θ at specified values. In this way we can obtain maximum likelihood estimates for the parameters of all the sub-models, as well as perform likelihood ratio tests for hypotheses concerning these sub-models. This will become clearer in the examples below.

An *S-Plus* library, called REIDAR (=REpaIrable systems Data Analysis Routines), containing routines for analysis of repairable systems, has been developed (Elvebakk, 1999).

In particular, the program contains routines for graphical analyses, parametric estimation in HTRP and the corresponding sub-models, trend testing and model checking.

Parametric estimation in HTRP models using REIDAR consists of maximising (10) by numerical integration and maximization. In most cases standard *S-plus* functions for integration (adaptive 15 point Gauss-Kronod quadrature) and minimization (quasi Newton) are used.

The parametric distributions used for F and H are Weibull, gamma and the bimodal exponential distribution (i.e. mixture of two exponential distributions). Recall that both F and H are required to have expected value 1. This is the explanation for the following somewhat unusual parameterizations. Density functions are given below. The corresponding distribution functions are denoted by capital letters, F_w etc.

Weibull: $f_w(x|\beta) = \beta^{-1}(\Gamma(\beta+1))^{\frac{1}{\beta}} x^{\frac{1}{\beta}-1} e^{(-(\Gamma(\beta+1)x)^{\frac{1}{\beta}})} \qquad \beta > 0$

Gamma: $f_g(x|\gamma) = \left(\Gamma(\frac{1}{\gamma})\right)^{-1} \gamma^{-\frac{1}{\gamma}} x^{\frac{1}{\gamma}-1} e^{-\frac{x}{\gamma}} \qquad \gamma > 0$

Bimodal exponential: $f_{be}(x|p,q) = p\alpha_1 e^{-\alpha_1 x} + (1-p)\alpha_2 e^{-\alpha_2 x} \qquad 0 < p, q < 1$
where $\alpha_1 = \frac{p(q-1)+1}{q}$ and $\alpha_2 = p(q-1)+1$

The last distribution was included motivated by a particular application (Section 8.3). A random variable from this distribution will with probability p be exponentially distributed with parameter α_1, and with probability $1-p$ be exponentially distributed with parameter α_2. Note that $q = \frac{\alpha_2}{\alpha_1}$ where $\alpha_2 < \alpha_1$.

The functions included as options for the trend function (aging intensity) $\lambda(\cdot)$ are:

Homogeneous trend: $\lambda_h(t|a) = a \quad a > 0$

Power law trend: $\lambda_{pl}(t|a,b) = abt^{b-1} \quad a, b > 0$

Log linear trend: $\lambda_{ll}(t|a,c) = ae^{ct} \quad a > 0, \ -\infty < c < \infty$

Log linear power law trend: $\lambda_{llpl}(t|a,b,c) = abt^{b-1}e^{ct} \ a, b > 0 \ -\infty < c < \infty$

Linear trend: $\lambda_l(t|d,e) = (d+et) \vee 0 \quad -\infty < d, e < \infty$

It is generally possible to freely choose trend functions and density functions for the renewal and heterogeneity distributions. However, as the optimization procedure is based on numerical integration and minimization methods, there is no guarantee that the real maximum is found. In those cases where we could compare the results with results from other methods (analytical or numerical) we have found the procedure to be reliable.

6 Model Checking

We base our model checking procedures mainly on the so-called residual process(es) (Andersen *et al.*, 1993) of a fitted model, i.e. the observed failure times transformed by the estimated cumulative conditional intensity.

From the definition of a TRP process we have that if $T_1, T_2, \cdots$ are generated by a TRP$(F, \lambda(\cdot))$ process then

$$\Lambda(T_1), \Lambda(T_2), \cdots \text{ is RP}(F, 1)$$

It also follows from general theory of point processes (Brémaud, 1981) that the process of integrated *conditional* intensities

$$\Lambda(T_1|\mathcal{F}_{T_1-}), \Lambda(T_2|\mathcal{F}_{T_2-}), \cdots \text{ is HPP}(1)$$

This is her seen directly from

$$\begin{aligned} \Lambda(T_i|\mathcal{F}_{T_i-}) - \Lambda(T_{i-1}|\mathcal{F}_{T_{i-1}-}) &= \int_{T_{i-1}}^{T_i} \lambda(t|\mathcal{F}_{t-})dt \\ &= \int_{T_{i-1}}^{T_i} z(\Lambda(t) - \Lambda(T_{i-1}))\lambda(t)dt \\ &= \int_0^{\Lambda(T_i)-\Lambda(T_{i-1})} z(y)dy \end{aligned} \tag{11}$$

which is exponential with parameter 1 since $\Lambda(T_i) - \Lambda(T_{i-1})$ has distribution F and $z(\cdot)$ is the hazard rate of F.

Thus either process $\{\Lambda(T_i)\}$ or $\{\Lambda(T_i|\mathcal{F}_{T_i-})\}$ may form the basis for model checking, and in fact they are equivalent, as (11) shows. Since the latter process is theoretically an HPP(1) and thus does not depend on model parameters, we shall find it most convenient to use this process in the following and call it the *residual process* when estimates are substituted for the model parameters.

Consider now the setting of subsection 4.2, where data are modelled as HTRP$(F, \lambda(\cdot), H)$. If the values a_j are given, then using (8) we can define for system j an HPP(1) with events at $R_{1j}, R_{2j}, \ldots$ where

$$R_{ij} = \int_0^{a_j\Lambda(T_{ij})} z(y)dy$$

Model checking will now be based on the corresponding residual processes, $\{\hat{R}_{ij}\}$. These are obtained in parametric TRP models by substituting the maximum likelihood estimates $\hat{z}(\cdot)$ and $\hat{\Lambda}(\cdot)$ for $z(\cdot)$ and $\Lambda(\cdot)$, respectively. For the heterogeneous models where the a_j are unobserved, we use the Bayes approach of Follmann & Goldberg (1988) in order to obtain estimates of the a_j: For the jth system, modelled by TRP$(F, a_j\lambda(\cdot))$, suppose a_j is the only unknown parameter, with given prior distribution H. The likelihood function is now $L_j(a_j)$ in (9), so the posterior distribution of a_j is

$$\Pi(a_j|T_{j1}, T_{j2}, \cdots, T_{jN_j}) \propto h(a_j)L_j(a_j) \tag{12}$$

The expectation of this posterior distribution, with estimated parameters used for F, $\lambda(\cdot)$ and H is used as our estimate of the heterogeneity parameter a_j.

In the examples of Section 8 we demonstrate how the residual processes $\{\hat{R}_{ij}\}$ are used in model checking. Typical properties to check will be the distribution of the interarrival times, i.e. the $\hat{R}_{ij} - \hat{R}_{i-1,j}$ (these will sometimes be called the *residuals*), the presence of trend (e.g. whether the residuals tend to increase or decrease with i), or serial correlation of the residuals.

Simulations have shown that the suggested method for estimation of the a_j has a tendency to "regression towards the mean". This may lead bias in the estimation of the a_j, which in turn will have consequences for the residual processes. This effect naturally diminishes as the number of failures in each system increase and the estimates improve.

7 Special Cases with Analytic Solution of Likelihood

7.1 Gamma Distributed Renewals and Heterogeneity

We assume now that m systems are observed under failure truncation, i.e. when the jth system is observed until n_j failures have occurred. In this case $\tau_j = T_{n_j j}$ so that the last factor of (9) disappears.

With notation as in the previous sections, let $F \equiv F_g$ be a gamma distribution with parameter γ, and let $\lambda(\cdot)$ for the moment be unspecified. For simplicity of writing, put $K_{ij} = \Lambda(T_{ij}) - \Lambda(T_{i-1,j})$ where $T_{0j} = 0$. Then we have from (9),

$$L_j(a_j) = \{\prod_{i=1}^{n_j} K_{ij}^{1/\gamma-1}\lambda(T_{ij})\}\frac{a_j^{n_j/\gamma}}{(\Gamma(1/\gamma))^{n_j}} \exp\{-a_j\Lambda(T_{n_j j})/\gamma\} \tag{13}$$

If there is no heterogeneity, i.e. the a_j are set to 1, we get the model TRP$(F_g, \lambda(\cdot))$. Further restricting to $\gamma = 1$ corresponds to the model NHPP$(\lambda(\cdot))$.

In case of heterogeneity, modelled by letting $H \equiv H_g$ be a gamma distribution with parameter δ, we find L_j by taking the expected value of $L_j(a_j)$ w.r.t. a_j, which here can be computed analytically to be,

$$L_j = \{\prod_{i=1}^{n_j} K_{ij}^{1/\gamma-1}\lambda(T_{ij})\}\frac{\gamma^{-n_j/\gamma}}{(\Gamma(1/\gamma))^{n_j}}\frac{\delta^{-1/\delta}}{\Gamma(\delta)}\frac{\Gamma(n_j/\gamma + 1/\delta)}{(1/\delta + (1/\gamma)\Lambda(T_{n_j j}))^{n_j/\gamma+1/\delta}}$$

Thus the model HTRP$(F_g, \lambda(\cdot), H_g)$ can be estimated by maximising $L = \prod L_j$ with respect to γ, δ and the parameters of $\lambda(\cdot)$.

Recall that for model checking we need to compute the posterior expectations of the a_j. In the present case it is easy to see that the posterior distribution is a gamma distribution, the expected value of which is given by

$$\frac{\gamma + n_j\delta}{\gamma + \delta\Lambda(T_{n_j j})} \tag{14}$$

This will be our estimate $\hat{a}_j$ of a_j when estimates are substituted for δ, γ and $\Lambda(\cdot)$.

It is interesting to note that we can write (14) as a convex combination of 1, which corresponds to the value of a_j we would have in the case of no heterogeneity, and $n_j/\Lambda(T_{n_j j})$ which is the natural estimator of a_j based solely on the jth system, when $a_j\Lambda(T_{1j}), a_j\Lambda(T_{2j}), \ldots$ is a renewal process with mean interval length equal to 1 (which is the basic assumption of the TRP).

7.2 Weibull Distributed Renewals, Gamma Distributed Heterogeneity

Again we assume that m systems are observed, where the jth system is observed in the interval $(0, \tau_j]$. However, now we shall allow the case where $T_{n_j j} < \tau_j$, which occurs for example in the case of time truncation censoring.

By convention, let $T_{0j} = 0$, $T_{n_j+1,j} = \tau_j$, and define K_{ij} as in the previous subsection. Then

$$L_j(a_j) = \{\prod_{i=1}^{n_j} f(a_j K_{ij}) a_j \lambda(T_{ij})\}\{1 - F(a_j K_{n_j+1,j})\}$$

With the Weibull density and distribution function with parameter β substituted for f and F, respectively, we get (where, $\alpha \equiv \Gamma(\beta+1)$),

$$L_j(a_j) = \beta^{-n_j}\alpha^{n_j/\beta}\{\prod_{i=1}^{n_j} K_{ij}^{1/\beta-1}\lambda(T_{ij})\}a_j^{n_j/\beta}\exp\{-\alpha^{1/\beta}(\sum_{i=1}^{n_j+1} K_{ij}^{1/\beta})a_j^{1/\beta}\} \qquad (15)$$

Again, put $a_j = 1$ in case of no heterogeneity, and observe that if in addition we put $\beta = 1$ (and hence $\alpha = 1$) we get the well known likelihood of an NHPP($\lambda(\cdot)$).

In the case of heterogeneity, modelled by a density $h(a)$, say, for a_j we need to integrate $L_j(a_j)h(a_j)$ with respect to a_j. In order to solve the integral explicitly we suggest using a generalized gamma distribution (Johnson, Kotz & Balakrishnan, 1994-1995) with density

$$h(a) \propto a^r e^{-(1/s)a^\beta} \qquad (16)$$

where β is the same parameter as is used for the renewal Weibull distribution F, $s > 0$ is a free parameter to be chosen, and $r \equiv \phi(s, \beta)$ is determined so that the expected value equals 1. Note that $s \to 0$ will then correspond to *no* heterogeneity. It is easy to compute the likelihood L_j explicitly. Note that Follmann & Goldberg (1988), with a different parametrization of the Weibull distribution, essentially uses this approach for the case of a Weibull renewal process with heterogeneity. It should also be noted, however, that our parametrization involves the unpleasant use of β as a parameter both in the renewal distribution and the heterogeneity distribution. This is of course not necessary using the computer program REIDAR, which utilizes numerical integration (see examples in next section).

Finally, the posterior expectation of the a_j, as needed for model checking, is straightforward to compute also in this case. This follows since the desired posterior distribution again will be a generalized gamma distribution, for which the expected value has a closed form expression.

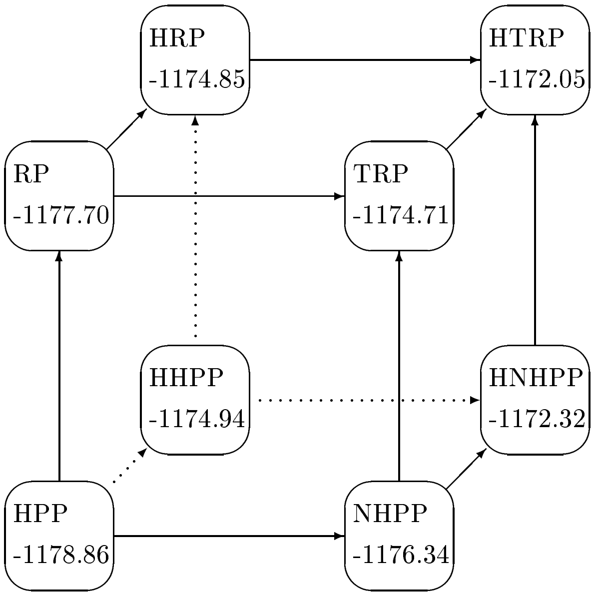

Figure 3: The log-likelihood cube of the HTRP$(F_w, \lambda_{pl}(\cdot), H_g)$ model for Proschan's air-conditioning failures data

8 Examples

8.1 Proschan's Data

As our first illustration we consider the very well-known data of air-plane air-conditioning failures given by Proschan (1963). There were originally data from thirteen planes, but since four of the planes had major overhauls for the air conditioning equipment, we let a new system start at the time of overhaul (Proschan, 1963) (Follmann & Goldberg, 1988). We thus have $m = 17$ systems with a total of 213 failures, where the number of failures for each systems varies from $n = 3$ to $n = 30$, all systems are failure truncated.

A HTRP$(F_w, \lambda_{pl}(\cdot), H_g)$ model has been fitted to this dataset by the numerical methods of Section 5. (For this model the integral of (10) can be solved analytically, see Section 7.2.) The log-likelihood cube including the seven sub-models is shown in Figure 3.

The conclusion of Proschan (1963) was that the data follow heterogeneous HPPs. Follmann & Goldberg (1988) tried a gamma-heterogeneous Weibull distributed RP, a HRP(F_w, λ, H_g) model in our notation, but concluded that a gamma heterogeneous HPP (HHPP(λ, H_g)) seemed like a better model. From the cube in Figure 3 a tentative conclusion may be that a gamma-heterogeneous power law NHPP (HNHPP$(\lambda_{pl}(\cdot), H_g)$) is a slightly better fitting model.

As Follmann & Goldberg (1988) pointed out the gamma heterogeneous HPP seems to give a better fit than a HPP. Two times the difference in the log-likelihood is 7.84, clearly significant. (The likelihood ratio statistic is asymptotically chi-squared distributed with

Model	Parameters	Estimates
$\mathrm{HHPP}(\lambda, H_g)$	(a, γ)	0.0108, 0.107
$\mathrm{HNHPP}(\lambda_{pl}(\cdot), H_g)$	(a, b, γ)	0.00343, 1.162, 0.114

Table 1: Parameter estimates for models $\mathrm{HHPP}(\lambda, H_g)$ and $\mathrm{HNHPP}(\lambda_{pl}(\cdot), H_g)$ for air-conditioning failures data.

one degree of freedom, with probability 0.5, and zero with probability 0.5 (Follmann & Goldberg, 1988).) Exchanging the HPP(1) in the above model with a Weibull renewal process only gives an insignificant increase in the log-likelihood. Introducing a power law trend function ($\mathrm{HNHPP}(\lambda_{pl}(\cdot), H_g)$), however, leads to an increase in two times the the log-likelihood of 5.24 compared to the $\mathrm{HHPP}(\lambda, H_g)$ model, giving a p-value of 0.022 assuming χ^2_1-distribution. We thus conclude that, based on the difference in log-likelihood alone, the gamma-heterogeneous power law Poisson process seems to be the best fitting model among those considered here.

To assess the model fit the estimated "residual" processes are used, given by $\hat{\Lambda}(T_{ji}|\mathcal{F}_{T_{ji}-})$ for $j = 1, \ldots, 17$, $i = 1, \ldots, N_j$. This results in 17 systems with the same number of failures as the original data. If the model is correct they should all follow HPPs with unit intensity (HPP(1)). To compare the two models $\mathrm{HHPP}(\lambda, H_g)$ and $\mathrm{HNHPP}(\lambda_{pl}(\cdot), H_g)$ the residual processes have been computed. These should be examined for deviations from HPP(1), like deterministic (global) trend, heterogeneity, serial correlation and distribution of the interarrival times. Akersten (1991) advocates the use of *total time on test* plotting techniques for these tasks, we will partly follow his recommendations. It should be noted that due to the low number of failures for some of the systems the heterogeneity parameters may be significantly biased.

Figure 4 shows the empirical distribution functions of the interarrival times of the residual processes plotted against the distribution function of the unit exponential distribution. If the fit is good the points should lie close to the diagonal, and consequently the plots seem to indicate a reasonably good fit for both models.

If the model is correct, residual processes from different systems should follow a common HPP. We may thus use the *total time on test* estimator for recurrent events (Barlow & Davis, 1977) to check for any (global) time trends. The plots for the residual processes of both models are shown in Figure 5. If there are no trends the points should again lie on the diagonal, a concave (convex) plot indicates monotonically increasing (decreasing) intensity. From Figure 5 there seem to be indications of increasing intensity in the residuals of the $\mathrm{HHPP}(\lambda, H_g)$ model, while the residuals of the $\mathrm{HNHPP}(\lambda_{pl}(\cdot), H_g)$ model have a closer fit to the HPP model.

This conclusion is also supported by trend tests. Again, since we want to check whether the residual processes follow a common HPP, we may use the TTT transform of the residual

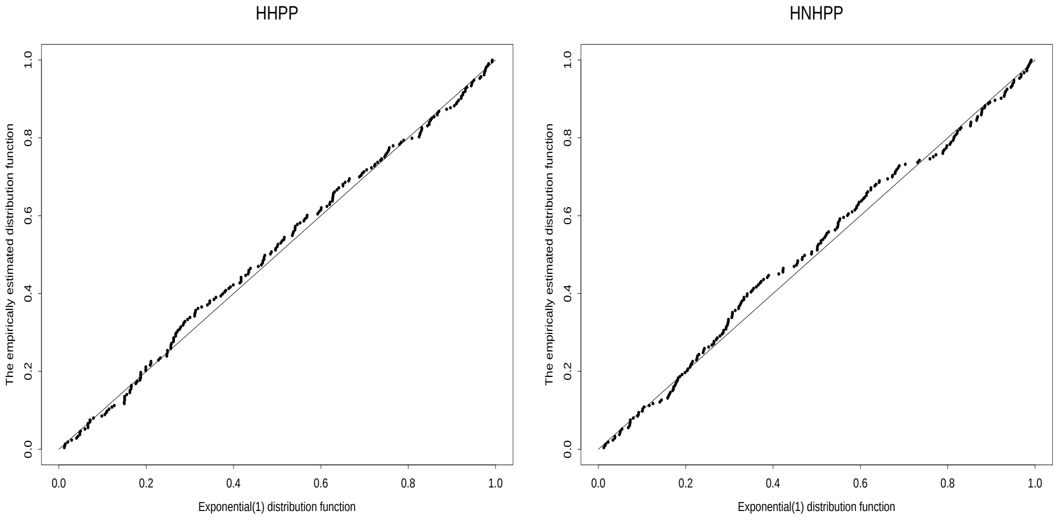

Figure 4: The distribution function for a unit exponential distribution plotted against the empirical distribution function of the interarrival times of the residual processes for models HHPP(λ, H_g) and HNHPP($\lambda_{pl}(\cdot), H_g$) for air-conditioning failures data.

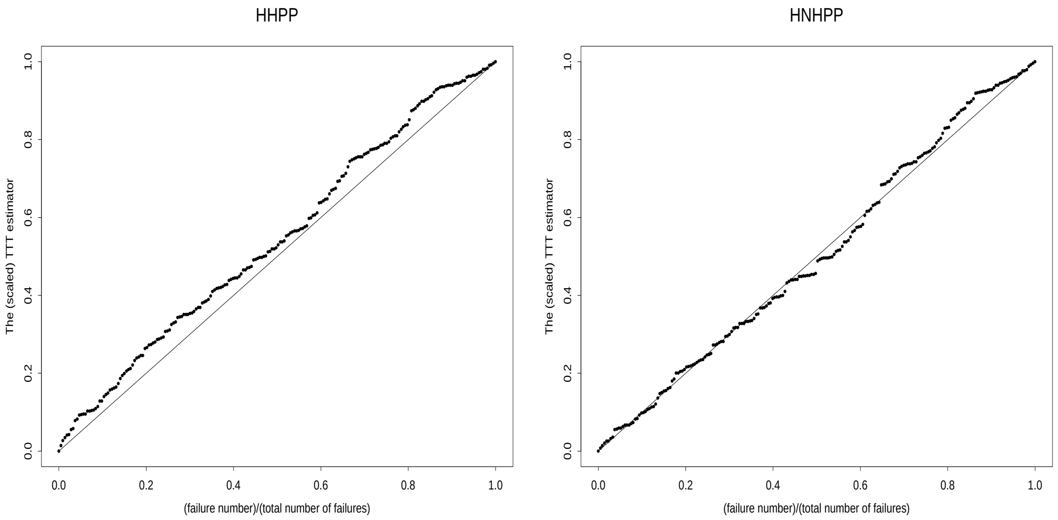

Figure 5: The recurrent events total time on test plot for the residuals of the HHPP(λ, H_g) and HNHPP($\lambda_{pl}(\cdot), H_g$) models for air-conditioning failures data.

processes as input to the tests (Kvaløy & Lindqvist, 1998). The so-called military handbook test for trend for a failure truncated system (Ascher & Feingold, 1984) is then

$$MH = 2\sum_{k=1}^{N-1} \ln\left(\frac{B-A}{\mathcal{T}_k - A}\right) \tag{17}$$

where A and B are the times of observation start and stop of the TTT transformed dataset, $\mathcal{T}_k$ is the k'th event time of the TTT transformed dataset and N is the total number of events of the dataset. This test statistic is χ^2_{2N}-distributed under the null hypothesis of common HPPs. For the residual processes of the HHPP(λ, H_g) and HNHPP$(\lambda_{pl}(\cdot), H_g)$ models the test statistics are 353.3 and 412.1, respectively, with 424 degrees of freedom. The corresponding two-sided p-values against trend are then 0.01 and 0.70. The conclusion is the same as indicated in the TTT plot: the residual processes of the HHPP(λ, H_g) model do not follow a common HPP, possibly is there an increasing intensity.

A test for more general deviations from HPP is the so-called Anderson-Darling test (Kvaløy & Lindqvist, 1998) (Lewis, 1963).

$$AD = -\frac{1}{N}\left[\sum_{k=1}^{N}(2k-1)(\ln\left(\frac{\mathcal{T}_k - A}{B-A}\right) + \ln(1 - \frac{\mathcal{T}_{N+1-k} - A}{B-A}))\right] - N + 1 \tag{18}$$

The null hypothesis is again a common HPP. For the residual processes of the HHPP(λ, H_g) and HNHPP$(\lambda_{pl}(\cdot), H_g)$ models the test statistics are 3.07 and 0.73, respectively. Using the asymptotic distribution of the Anderson-Darling test statistic (Anderson & Darling, 1954) the corresponding p-values are 0.025 and 0.53. Again a reasonable conclusion is that the residual processes of the HHPP(λ, H_g) model do not follow a (common) HPP.

Going back to the distribution of the interarrival times of the residual processes, the same tests may also be used here. The TTT estimator is now the standard TTT estimator of independent and identically distributed data, i.e. the interarrival times of the residual processes. The Anderson-Darling test statistic is the same as in (18), but this is now a test for non-exponential distribution of the interarrival times. For the interarrival times of the residual processes of the HHPP(λ, H_g) and HNHPP$(\lambda_{pl}(\cdot), H_g)$ models the test statistics are 0.66 and 1.12, respectively. The corresponding p-values are 0.59 and 0.30. This supports the conclusions drawn from the plots of Figure 4: there is no reason to reject the null hypothesis of an exponential distribution.

To detect a possible serial dependence in the residual processes we compute an estimate of the auto correlation function for lag k:

$$\rho(k) = \frac{1}{m}\sum_{j=1}^{m}\left[\frac{N_j}{N_j - k}\frac{\sum_{i=1}^{N_j-k}(X_{i+k} - \overline{X})(X_i - \overline{X})}{\sum_{i=1}^{N_j}(X_i - \overline{X})^2}\right] \tag{19}$$

The estimated asymptotic variance of $\rho(k)$ is $\frac{1}{m^2}\sum_{j=1}^{m}\frac{1}{N_j}$. For the residual processes of the two models the estimated auto correlation functions at lag 1 are -0.13 and -0.09, respectively. In both cases the estimated standard deviation is 0.09, so assuming normality

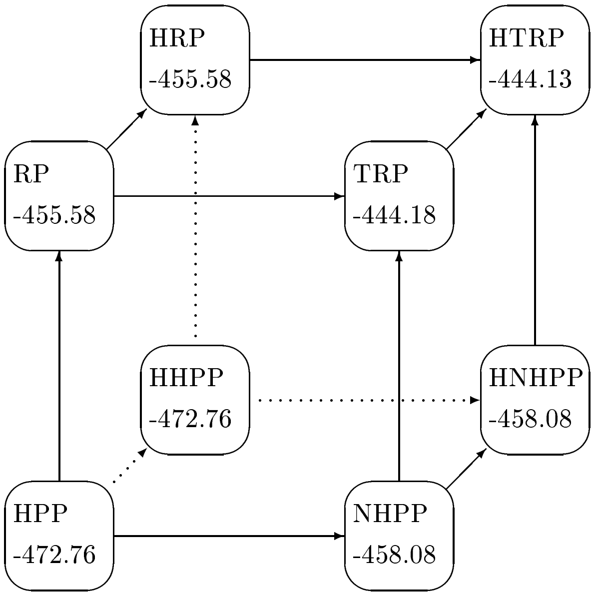

Figure 6: The log-likelihood cube of the HTRP($F_g, \lambda_{pl}(\cdot), H_g$) model for the tractor engine failures data

we cannot reasonably conclude that there is serial correlation at lag 1 in either dataset. The auto correlation function estimates for higher lags were also found non significant.

To sum up, in the HTRP($F_w, \lambda_{pl}(\cdot), H_g$) class of models for Proschan's air-plane air-conditioning data a HNHPP($\lambda(\cdot)_{pl}, H_g$) model seems to give a better fit than a HHPP(λ, H_g) model. This is based on difference in the log-likelihood of the two models, and supported by the "residual" analysis: Although the interarrival times of the HHPP(λ, H_g) model seem to be unit exponentially distributed there are indications of a possible increasing intensity in (global) time.

8.2 Tractor Data

This is a dataset from Barlow & Davis (1977) of engine replacement times (in operating hours) for 22 tractors. Originally these data were failure truncated, but since very few systems survived towards the end of the observation period we chose to censor all systems at 15000 hours. We thus have $m = 22$ systems with a total of 50 failures, where the number of failures for each system varies from $n = 1$ to $n = 4$. Four of the systems are time truncated, the rest are failure truncated.

A HTRP($F_g, \lambda_{pl}(\cdot), H_g$) model has been fitted to this dataset by the numerical methods of Section 5. (For this model with failure truncated systems the integral of (10) can be solved analytically, Section 7.1.) The resulting log-likelihood cube including the seven sub-models is shown in Figure 6.

Starting from the homogeneous Poisson process model (HPP(λ)) there is a large gain

Model	Parameters	Estimates
$\text{RP}(F_g, \lambda)$	(γ, a)	0.271, 0.000219
$\text{NHPP}(\lambda_{pl}(\cdot))$	(a, b)	$2.671 \cdot 10^{-9}$, 2.205
$\text{TRP}(F_g, \lambda_{pl}(\cdot))$	(γ, a, b)	0.258, $2.006 \cdot 10^{-6}$, 1.504

Table 2: Parameter estimates for models $\text{RP}(F_g, \lambda)$, $\text{NHPP}(\lambda_{pl}(\cdot))$ and $\text{TRP}(F_g, \lambda_{pl}(\cdot))$ for tractor engines failures data.

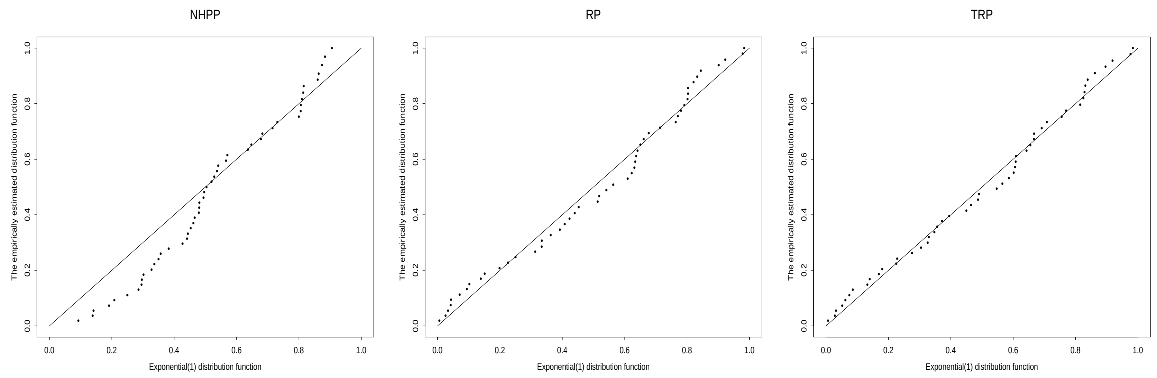

Figure 7: The distribution function for a unit exponential distribution plotted against the Kaplan-Meier empirical distribution function of the interarrival times of the residual processes for models $\text{RP}(F_g, \lambda)$, $\text{NHPP}(\lambda(\cdot)_{pl})$ and $\text{TRP}(F_g, \lambda_{pl}(\cdot))$ for tractor engine failures data.

in log-likelihood when going either to a gamma renewal process ($\text{RP}(F_g, \lambda)$) or a power law non-homogeneous Poisson process ($\text{NHPP}(\lambda_{pl}(\cdot))$). Both are clearly better models judged by the difference in log-likelihood. The same is true with regard to the gamma power law trend renewal process ($\text{TRP}(F_g, \lambda_{pl}(\cdot))$) compared to the $\text{RP}(F_g, \lambda)$ or $\text{NHPP}(\lambda(\cdot)_{pl})$ models. There are no significant gains in introducing gamma heterogeneity for either of these models. So, from the difference in log-likelihood the $\text{TRP}(F_g, \lambda_{pl}(\cdot))$ model seems to be a better fit than the $\text{RP}(F_g, \lambda)$ or the $\text{NHPP}(\lambda_{pl}(\cdot))$ models. To show the effects on the residual processes, these have been computed for all three models.

In Figure 7 the Kaplan-Meier empirical distribution functions of the interarrival times of the residual processes are plotted against the unit exponential distribution function. For both the $\text{RP}(F_g, \lambda)$ and $\text{TRP}(F_g, \lambda_{pl}(\cdot))$ models there are a reasonably good fit, while there seems to be some deviations for the $\text{NHPP}(\lambda_{pl}(\cdot))$ model. Anderson-Darling tests for exponential distribution, equivalent to the test of Section 8.1, give test statistics of 4.84, 0.75 and 0.31 for the $\text{NHPP}(\lambda_{pl}(\cdot))$, $\text{RP}(F_g, \lambda)$ and $\text{TRP}(F_g, \lambda_{pl}(\cdot))$ models, respectively. The corresponding p-values are 0.003, 0.52 and 0.93. So an underlying renewal process (F_g) with standard deviation of about 0.5 fits the data better than a unit intensity HPP.

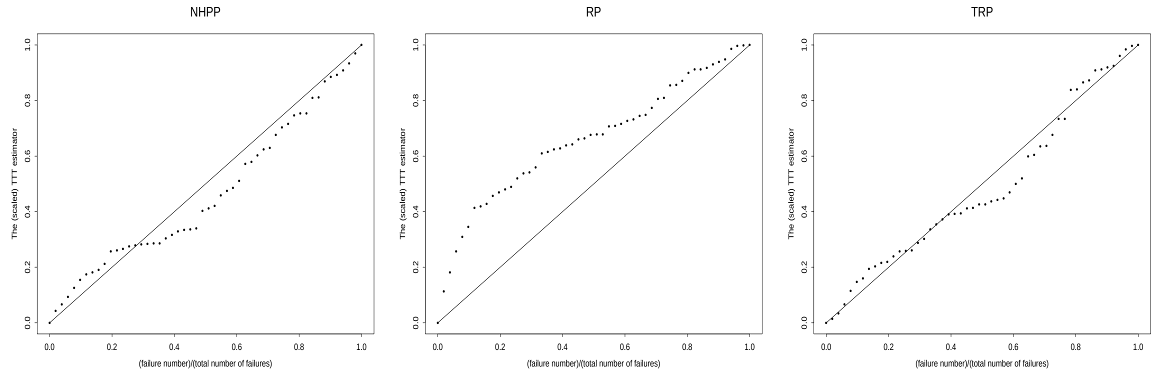

Figure 8: The recurrent events total time on test plot for the residuals of the NHPP($\lambda_{pl}(\cdot)$), RP(F_g, λ) and TRP($F_g, \lambda_{pl}(\cdot)$) models for tractor engine failures data.

From the recurrent events TTT plots in Figure 8 we observe that the residual processes of the RP(F_g, λ) model clearly deviate from a homogeneous Poisson process, while the TTT plots of the NHPP($\lambda_{pl}(\cdot)$) and TRP($F_g, \lambda(\cdot)_{pl}$) models show a reasonably good fit. The plot for the RP(F_g, λ) model indicates a possible increasing failure intensity, which is already incorporated in the other two plots and which explains why there is better model fit for these models.

These conclusions are again supported by trend tests. By using the same TTT type trend tests as in section 8.1 we get test statistics for the military handbook test of 96.5 (100 degrees of freedom) 48.6 (98) and 96.7 (100) for the NHPP($\lambda_{pl}(\cdot)$), RP(F_g, λ) and TRP($F_g, \lambda_{pl}(\cdot)$) models, respectively. The corresponding p-values against trend are 0.84, $6 \cdot 10^{-6}$, and 0.85. The conclusion is that the residual processes of the RP(F_g, λ) model do not follow a (common) HPP, there may be an increasing intensity.

Using the Anderson-Darling test (Section 8.1) we get test statistics of 1.19, 8.44 and 0.72 for the NHPP($\lambda_{pl}(\cdot)$), RP(F_g, λ) and TRP($F_g, \lambda_{pl}(\cdot)$) models, respectively. The corresponding p-values are 0.27, $7 \cdot 10^{-5}$ and 0.54. Again the conclusion must be that the residual processes of the RP(F_g, λ) model do not follow a (common) HPP.

To detect a possible serial dependence in the residual processes the auto correlation function estimator (19) is used. For the residual processes of the three models the estimated autocorrelation functions at lag 1 are 0.11, 0.13 and 0.26, for the RP(F_g, λ), NHPP($\lambda_{pl}(\cdot)$) and TRP($F_g, \lambda_{pl}(\cdot)$) models, respectively. The estimated standard deviation is 0.16, so we cannot reasonably conclude that there is serial correlation in either dataset, although the value for the TRP($F_g, \lambda_{pl}(\cdot)$) model is somewhat large.

To sum up, the differences in log-likelihood and analysis of "residual" processes indicates that 1) there is a trend present in the tractor engine data, and 2) the interfailure times are not consistent with the "bad as old" property of an NHPP. A possible physical interpretation of the failure mechanisms based on this is that the replacement of a major part (the engine) of the system leads to a renewal-type behaviour with a smaller variance than the HPP, while the general aging of the rest of the system introduces an increased

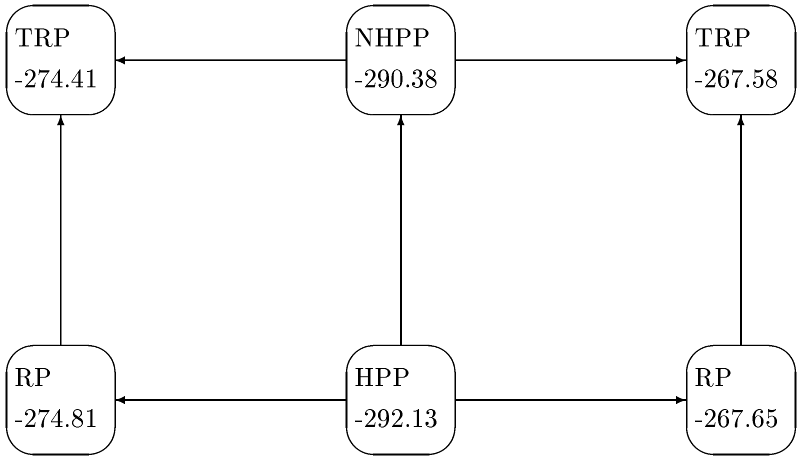

Figure 9: The log-likelihood grid of the TRP($F_w, \lambda_{pl}(\cdot)$) (left side) and TRP($F_{be}, \lambda_{pl}(\cdot)$) (right side) models for compressor failures data.

replacement frequency. Our conclusion is that for the tractor engine replacement data a TRP($F_g, \lambda_{pl}(\cdot)$) model seems to give the best fit among the HTRP($F_g, \lambda(\cdot)_{pl}, H_g$) class of models. Also the residual processes of this model seems to be consistent with a HPP(1) model.

8.3 Compressor Data

This dataset contains failure times (in days) of a gas compressor at Norsk Hydro in Porsgrunn, Norway. The full dataset includes failure times from a system of two compressors, from the start of the factory in 1968 until 1989 (Erlingsen, 1989). We have extracted the failures deemed critical from one of the compressors. We thus have $m = 1$ system with a total of 49 failures, censoring is here failure truncation.

Two TRP models: TRP($F_w, \lambda_{pl}(\cdot)$) and TRP($F_{be}, \lambda_{pl}(\cdot)$) have been fitted to this dataset by the numerical methods of Section 5. The resulting log-likelihood grid including the sub models is shown in Figure 9.

Basing our model choice on difference in log-likelihood we reject the HPP(λ) model in favour of the RP(F_w, λ) model with a p-value of $4 \cdot 10^{-9}$. Likewise the HPP(λ) model is rejected in favour of the RP(F_{be}, λ) model with a p-value of $2 \cdot 10^{-11}$. Note that the bimodal exponential RP model has 1 parameter more than the Weibull RP model and thus 2 parameters more than the HPP model. Only for the HPP(λ) model is there (weak) reason to include a power law trend function in the model. Judged by the difference in p-values, the bimodal exponential renewal process seems like a slightly better model than

Model	Parameters	Estimates
HPP(λ)	(a)	0.00700
RP(F_w, λ)	(γ, a)	1.835, 0.00633
RP(F_{be}, λ)	(p, q, a)	0.260, 0.00676, 0.00700

Table 3: Parameter estimates for models HPP(λ), RP(F_w, λ) and RP(F_{be}, λ) for compressor failure data.

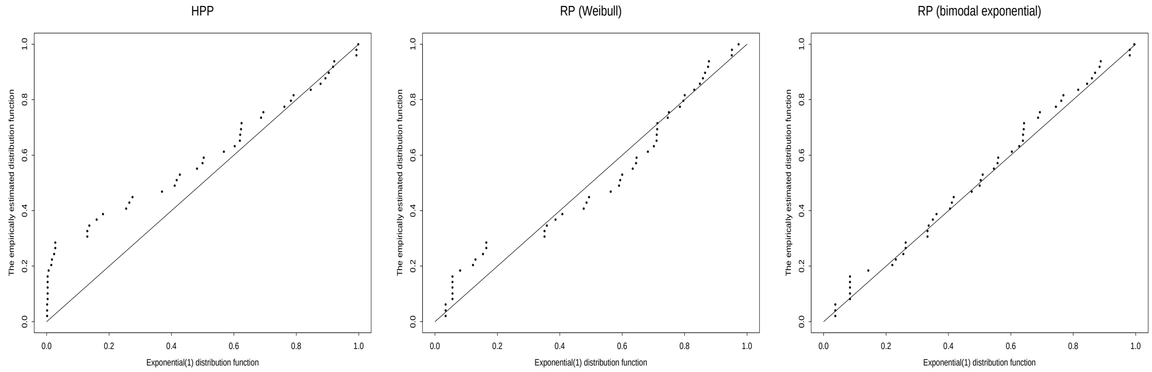

Figure 10: The distribution function for a unit exponential distribution plotted against the empirical distribution function of the interarrival times of the residual processes for models HPP(λ), RP(F_w, λ) and RP(F_{be}, λ) for compressor failures data.

the Weibull renewal process.

The empirical distribution of the interarrival times of the residual processes of the models HPP(λ), RP(F_w, λ) and RP(F_{be}, λ) are plotted against the distribution function of the unit exponential distribution in Figure 10. For the residuals of the RP(F_{be}, λ) model there is a very good fit, a bit worse for the RP(F_w, λ) model residuals, while the residuals of the HPP(λ) are clearly not unit exponentially distributed. This is supported by an Anderson-Darling test for exponential distribution, the test statistics are 13.83, 1.56 and 0.56 for the for the HPP(λ), RP(F_w, λ) and RP(F_{be}, λ) models, respectively. The corresponding p-values are $3 \cdot 10^{-7}$, 0.16 and 0.68, the residual process of the HPP(λ) model seems to be overdispersed compared to a (unit intensity) homogeneous Poisson process.

From the recurrent events *total time on test* plots in Figure 11 the residual processes of all three models are quite close to the diagonal. However, we see that the residuals of the RP(F_w, λ) model most closely resemble an HPP(1). The residuals of the RP(F_{be}, λ) model are also reasonably close to the diagonal, while the HPP(λ) residuals deviate somewhat from an HPP(1). As all three model incorporate the same trend function a possible explanation for this behaviour is that a renewal process overdispersed compared to an HPP(1) will depart more from the diagonal in a TTT plot.

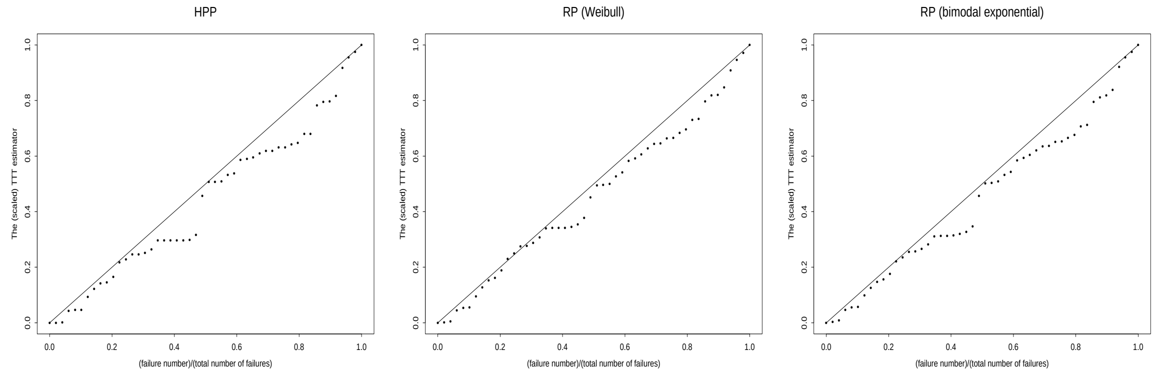

Figure 11: The recurrent events total time on test plot for the residuals of the HPP(λ), RP(F_w, λ) and RP(F_{be}, λ) models for compressor failures data.

By using the same (TTT type) trend tests as in Section 8.1 we get test statistics for the military handbook test of 126.6 (96 degrees of freedom), 111.5 (96) and 111.4 (96) for the HPP(λ), RP(F_w, λ) and RP(F_{be}, λ) models, respectively. The corresponding p-values against trend are 0.04, 0.26 and 0.26, reasonably strong evidence against a homogeneous Poisson process for the residual processes of the HPP(λ) model.

Using the Anderson-Darling test we get test statistics of 1.84, 0.91 and 1.11 for the for the HPP(λ), RP(F_w, λ) and RP(F_{be}, λ) models, respectively. The corresponding p-values are 0.11, 0.41 and 0.30.

To detect a possible serial dependence in the residual processes we compute the estimator for the auto correlation function (19). For the residual processes of the three models the estimated autocorrelation functions at lag 1 are -0.16, -0.16 and 0.11, for the HPP(λ), RP(F_w, λ) and RP(F_{be}, λ) models, respectively. The estimated standard deviation is 0.14, so we cannot reasonably conclude that there is serial correlation in either dataset.

To sum up, for these data both the RP(F_w, λ) and RP(F_{be}, λ) models fit reasonably well. Both of these models are overdispersed compared to an HPP(λ) model, meaning that the variance of F are greater than 1. Based on the difference in log-likelihood and the very good fit to the empirical distribution we tend to prefer the bimodal exponential renewal process, RP(F_{be}, λ). A possible interpretation of this model is that two types of repair can occur when the compressor fails. With probability $1 - p = 0.74$ the repair is successful and the time to next failure is governed by an exponential distribution with expectation $\frac{1}{(a(p(q-1)+1)} = 192.6$ days. And with probability $p = 0.26$ the repair is unsuccessful and the time to next failure is governed by an exponential distribution with expectation $\frac{q}{p(q-1)+1} =$ 1.30 days.

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