Trees, renormalisation and differential equations

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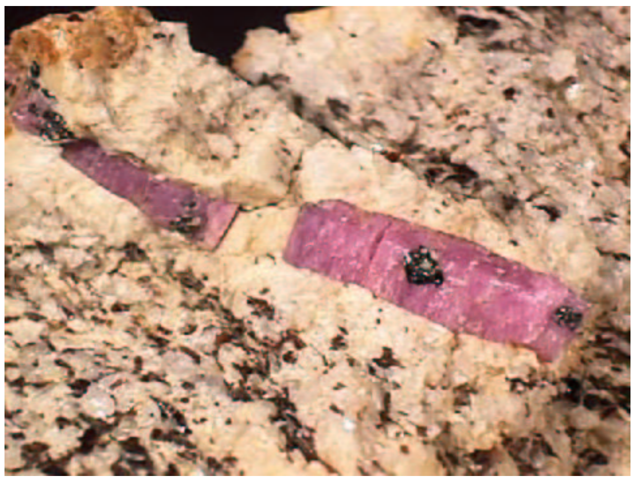
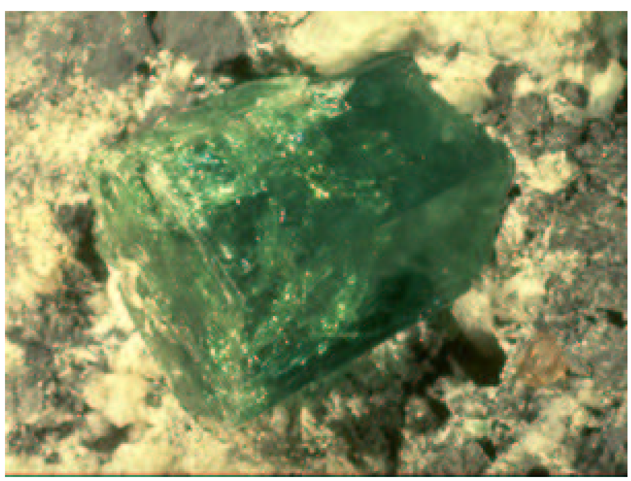




The Hopf algebra of rooted trees

- 1972 J.C. Butcher. *Algebra of Runge-Kutta methods*
- 1986 A. Dür. *Butcher's algebra is a Hopf algebra*
- 1998 D. Kreimer. *The Hopf algebra of renormalisation in quantum field theory*
- 1998 A. Connes and H. Moscovici. *Index theorem of the noncommutative geometry of foliations*

We regard Butcher's work on the classification of numerical integration methods as an impressive example that concrete problem-oriented work can lead to far-reaching conceptual results.

Alain Connes:Fields Medal (1982), Crafoord Prize (2001), Member of the Academy of Sciences of Norway (1993)

Layout of the talk

- The Hopf algebra of derivations
- Hopf algebras
- The Hopf algebra of rooted trees (Butcher)
- Runge-Kutta methods (Butcher)
- Renormalisation in quantum field theory (Kreimer, Connes, Wulkenhaar, Krajewski, Frabetti)
- The Connes-Moscovici algebra
- Composition of series
- Open questions

The Hopf algebra of derivations

The algebra

- $\mathcal{A}$: algebra of differential operators with constant coefficients
- Generators of $\mathcal{A}$: partial derivatives $\partial_i = \frac{\partial}{\partial x_i}$,
- Basis of $\mathcal{A}$: multiple partial derivatives $\frac{\partial^n}{\partial x_{i_1} \dots \partial x_{i_n}}$ for $n \geq 1$.
- Product: derivatives of derivatives $\partial_i \partial_j = \frac{\partial^2}{\partial x_i \partial x_j}$
- For any $D \in \mathcal{A}$, $D\mathbf{1} = \mathbf{1}D = D$.
- Example: $\frac{1}{\sqrt{2}}\mathbf{1} + \frac{\partial}{\partial x_1} - 4\frac{\partial^2}{\partial x_2 \partial x_3} \in \mathcal{A}$.
- $\mathcal{A}$ with the product of derivations is a unital associative algebra.

The Hopf algebra of derivations

The coproduct

- Action of the derivations on functions

$$\mathbf{1}(fg) = fg.$$

$$\partial_i(fg) = (\partial_i f)g + f(\partial_i g). \qquad \text{(Leibniz)}$$

$$\partial_i\partial_j(fg) = (\partial_i\partial_j f)g + (\partial_i f)(\partial_j g) + (\partial_j f)(\partial_i g) + f(\partial_i\partial_j g).$$

- $D(fg) = \sum (D_{(1)} f)(D_{(2)} g)$.
- $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad \Delta D = \sum D_{(1)} \otimes D_{(2)}$.
- Coproduct

$$\Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1}.$$

$$\Delta\partial_i = \partial_i \otimes \mathbf{1} + \mathbf{1} \otimes \partial_i.$$

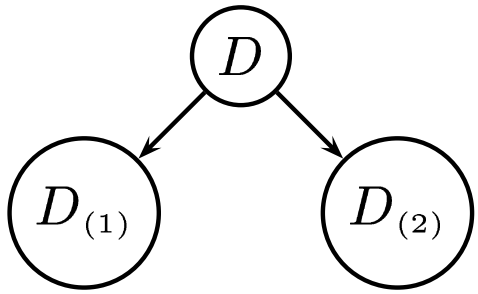
$$\Delta\partial_i\partial_j = \partial_i\partial_j \otimes \mathbf{1} + \partial_i \otimes \partial_j + \partial_j \otimes \partial_i + \mathbf{1} \otimes \partial_i\partial_j.$$

- $\Delta(DD') = (\Delta D)(\Delta D')$

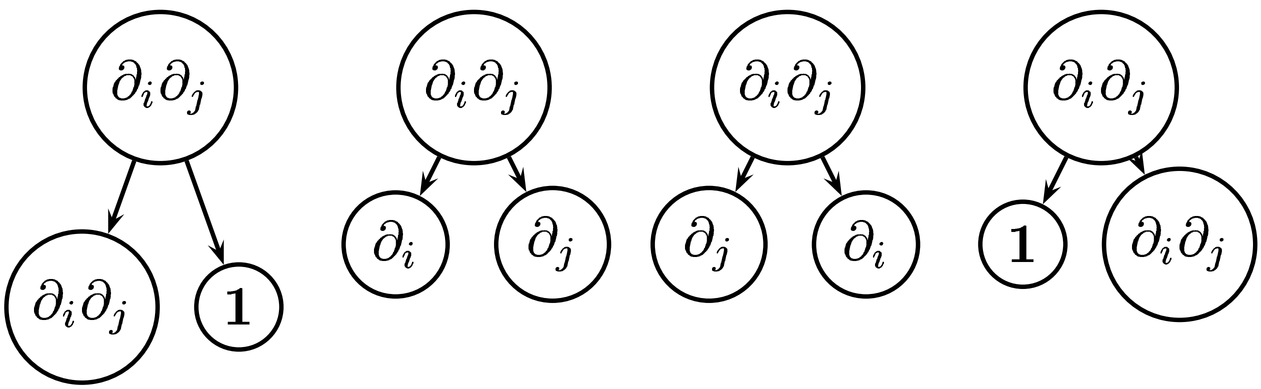
$$\Delta(\partial_i\partial_j) = (\partial_i \otimes \mathbf{1} + \mathbf{1} \otimes \partial_i)(\partial_j \otimes \mathbf{1} + \mathbf{1} \otimes \partial_j).$$

Coproduct: splitting into two parts

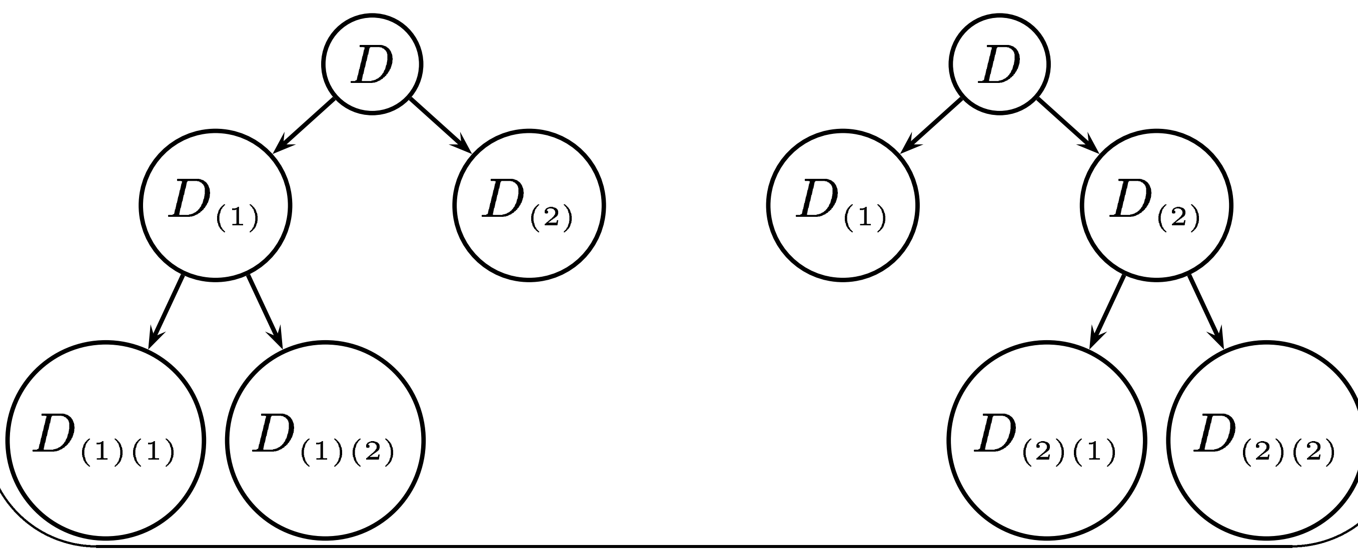
$$\Delta D = \sum D_{(1)} \otimes D_{(2)}$$



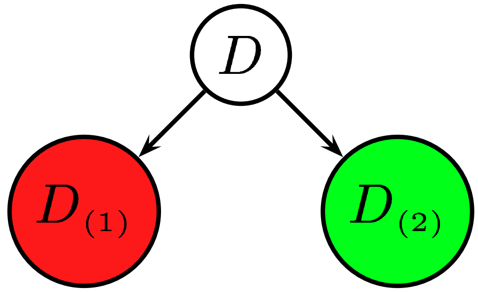
$$\Delta(\partial_i \partial_j) = \partial_i \partial_j \otimes \mathbf{1} + \partial_i \otimes \partial_j + \partial_j \otimes \partial_i + \mathbf{1} \otimes \partial_i \partial_j$$



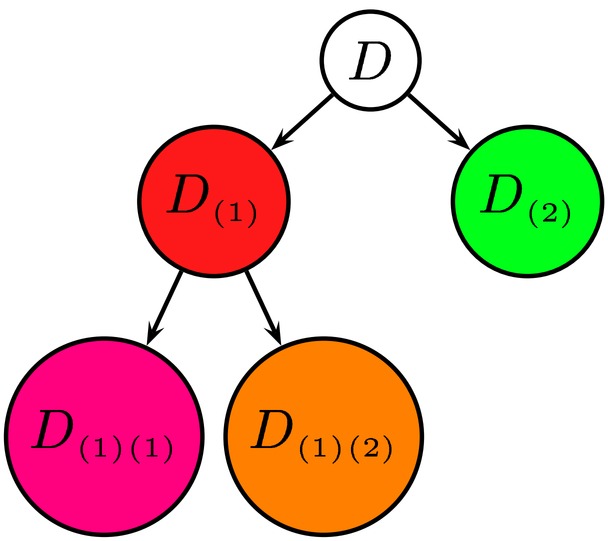
Splitting into three parts?

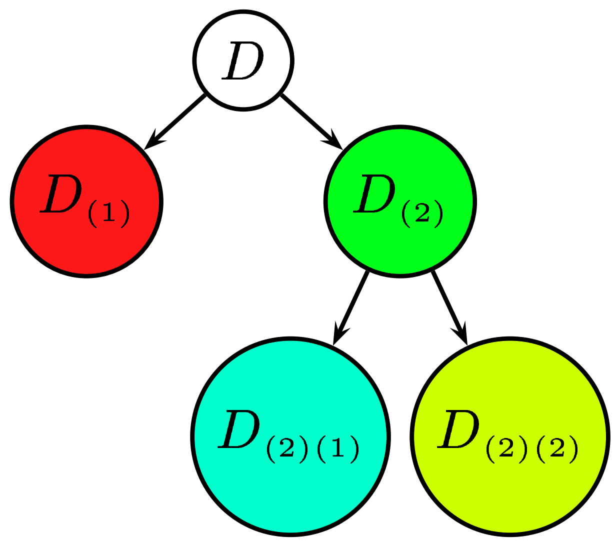


Coassociativity



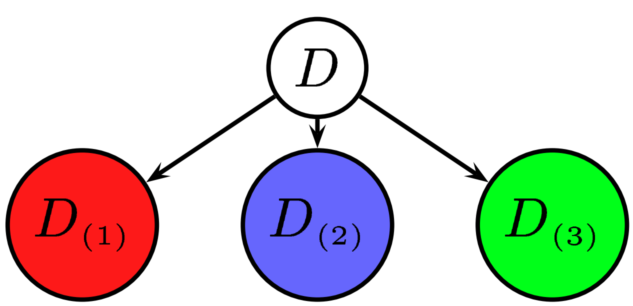
$$\Delta D = \sum D_{(1)} \otimes D_{(2)}$$





$$\sum (\Delta D_{(1)}) \otimes D_{(2)} \quad = \quad \sum D_{(1)} \otimes (\Delta D_{(2)})$$

$$(\Delta \otimes \mathrm{Id})\Delta D = (\mathrm{Id} \otimes \Delta)\Delta D = \sum D_{(1)} \otimes D_{(2)} \otimes D_{(3)}$$



The Hopf algebra of derivations

Coassociativity

- $D(fg) = \sum (D_{(1)} f)(D_{(2)} g)$.
- $fgh = fg\ h = f\ gh$.
- $D(fgh) = \sum (D_{(1)}(fg))(D_{(2)} h) = \sum (D_{(1)} f)(D_{(2)}(gh))$
  $= \sum (D_{(1)} f)(D_{(2)} g)(D_{(3)} h)$.
- $(\Delta \otimes \mathrm{Id})\Delta D = (\mathrm{Id} \otimes \Delta)\Delta D = \sum D_{(1)} \otimes D_{(2)} \otimes D_{(3)}$.
- Example $\Delta \partial_i = \partial_i \otimes \mathbf{1} + \mathbf{1} \otimes \partial_i$.

$$(\Delta \otimes \mathrm{Id})\Delta\partial_i = (\Delta\partial_i) \otimes \mathbf{1} + (\Delta\mathbf{1}) \otimes \partial_i$$
$$= \partial_i \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \partial_i \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \partial_i.$$
$$(\mathrm{Id} \otimes \Delta)\Delta\partial_i = \partial_i \otimes (\Delta\mathbf{1}) + \mathbf{1} \otimes (\Delta\partial_i)$$
$$= \partial_i \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \partial_i \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \partial_i.$$

- $D(f_1 \ldots f_n) = \sum (D_{(1)} f_1) \ldots (D_{(n)} f_n)$.

The counit

$$\varepsilon : \mathcal{A} \to \mathbb{C}, \quad D1 = \varepsilon(D)1$$

- $D = \mathbf{1}$: $\quad \mathbf{1}f = f$, so $\mathbf{1}1 = 1$ and $\varepsilon(\mathbf{1}) = 1$
- $\partial_i 1 = 0$ thus $\varepsilon(\partial_i) = 0$
- $\partial_{i_1} \ldots \partial_{i_n} 1 = 0$ thus $\varepsilon(\partial_{i_1} \ldots \partial_{i_n}) = 0$

$$\begin{aligned} Df &= D(1f) = \sum (D_{(1)}1)(D_{(2)}f) \\ &= \sum \varepsilon(D_{(1)})1(D_{(2)}f) = \sum \varepsilon(D_{(1)})(D_{(2)}f) \end{aligned}$$

Defining property of the counit:

$$D = \sum \varepsilon(D_{(1)})D_{(2)} = \sum D_{(1)}\varepsilon(D_{(2)})$$

The antipode

$$S : \mathcal{A} \to \mathcal{A}$$

Defining property of the antipode:

$$\sum S(D_{(1)})D_{(2)} = \sum D_{(1)}S(D_{(2)}) = \varepsilon(D)\mathbf{1}$$

- $S(\mathbf{1}) = \mathbf{1}$

  $$\Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1},$$

  $$S(\mathbf{1})\mathbf{1} = \varepsilon(\mathbf{1})\mathbf{1} = \mathbf{1},$$

- $S(\partial_i) = -\partial_i$

  $$\Delta\partial_i = \partial_i \otimes \mathbf{1} + \mathbf{1} \otimes \partial_i,$$

  $$S(\partial_i)\mathbf{1} + S(\mathbf{1})\partial_i = \varepsilon(\partial_i)\mathbf{1} = 0,$$

- $S(\partial_{i_1} \ldots \partial_{i_n}) = (-1)^n \partial_{i_1} \ldots \partial_{i_n}.$

Hopf algebra

- $\mathcal{A}$ is a unital associative algebra
- A coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\Delta a = \sum a_{(1)} \otimes a_{(2)}$

  Morphism: $\Delta(a \cdot b) = \sum (a_{(1)} \cdot b_{(1)}) \otimes (a_{(2)} \cdot b_{(2)})$

  Coassociativity: $(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta$
- A counit ε: $\mathcal{A} \to \mathbb{C}$,

  $\sum \varepsilon(a_{(1)}) a_{(2)} = \sum a_{(1)} \varepsilon(a_{(2)}) = a$

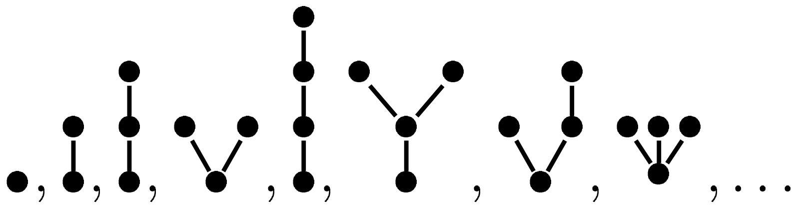
  Morphism: $\varepsilon(a \cdot b) = \varepsilon(a)\varepsilon(b)$
- An antipode S: $\mathcal{A} \to \mathcal{A}$,

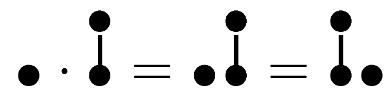
  $\sum S(a_{(1)}) a_{(2)} = \sum a_{(1)} S(a_{(2)}) = \varepsilon(a)\mathbf{1}$

  Antimorphism: $S(a \cdot b) = S(b) \cdot S(a)$.

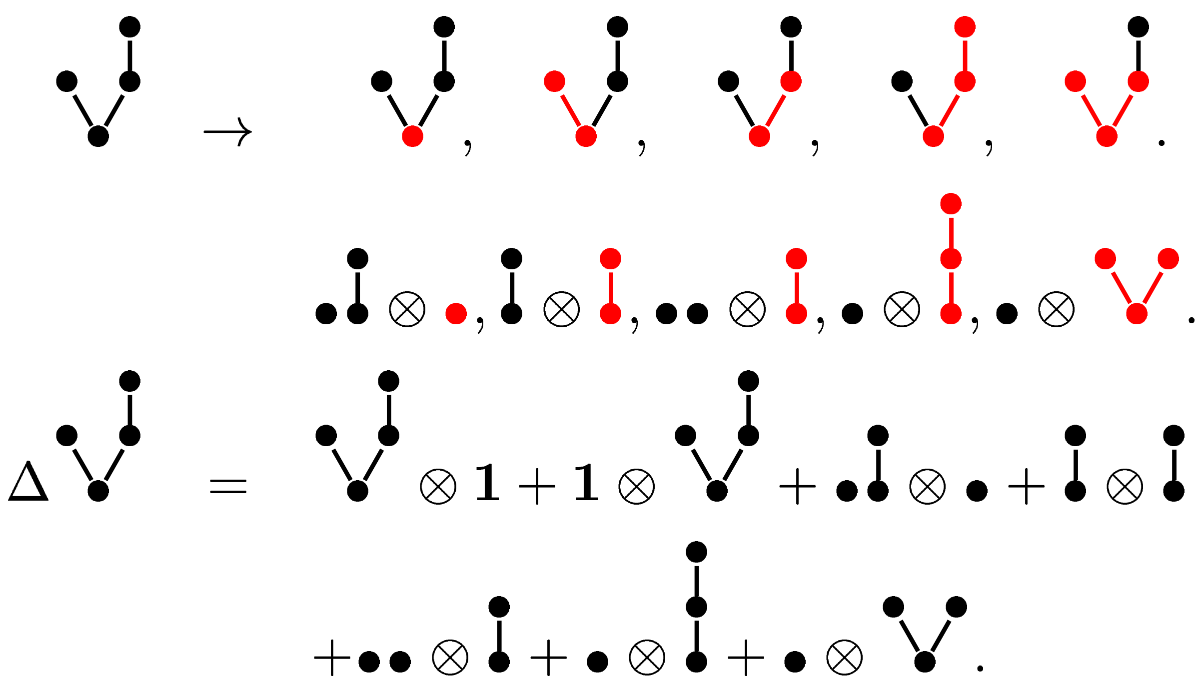
Butcher's Hopf algebra

- Algebra of rooted trees. Generated by $\mathbf{1}$ and



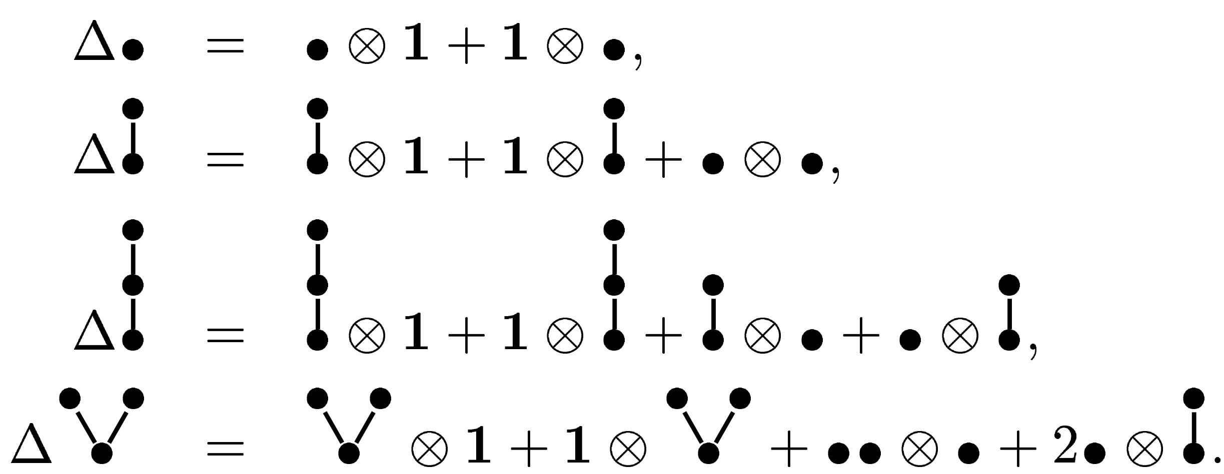
- Product: juxtaposition $t \cdot t' = tt'$. E.g. 

- Coproduct:
  $\Delta t = \sum t_{(1)} \otimes t_{(2)} = t \otimes \mathbf{1} + \mathbf{1} \otimes t + \sum' t_{(1)} \otimes t_{(2)}$, where $t_{(2)}$ is a proper subtree of t, such that $t_{(2)}$ is connected and shares the same root with t, and $t_{(1)}$ denotes the graph formed by deleting $t_{(2)}$ from t (Butcher, Miniature 15).



Butcher's Hopf algebra

- Examples of coproducts

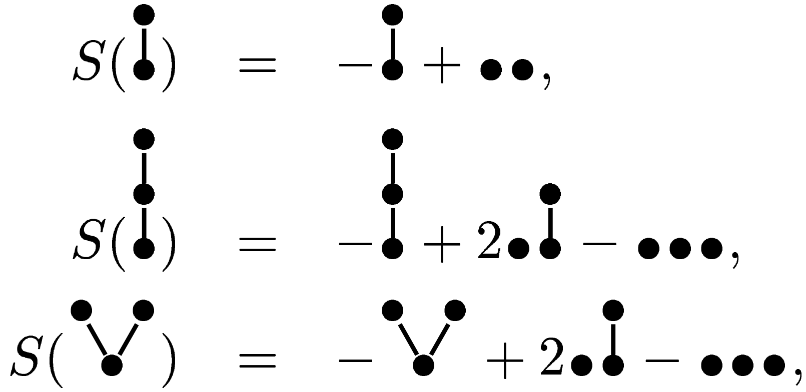


- Counit: $\varepsilon(\mathbf{1}) = 1$, $\varepsilon(t) = 0$
- Antipode: $\sum S(t_{(1)})t_{(2)} = \varepsilon(t)\mathbf{1}$

$$S(\mathbf{1}) = \mathbf{1},$$

$$S(t) = -t - {\sum}' S(t_{(1)})t_{(2)},$$

$$S(\bullet) = -\bullet,$$



Butcher's solution of differential equations

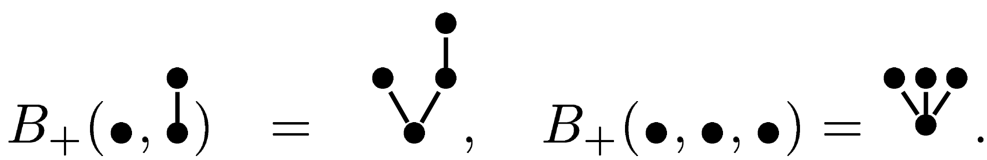
- The flow equation

$$\frac{\mathrm{d}x(s)}{\mathrm{d}s} = f(x(s)), \quad x(0) = x_0.$$

- Butcher's solution (1963)

$$x(s) = x_0 + \sum_t \frac{s^{|t|}}{\sigma(t)} \frac{1}{t!} \delta_t$$

- The sum is over all rooted trees t
- $|t|$= number of vertices of t
- Tree merger B_+: $B_+(t_1, \ldots, t_n)$ links the trees $t_1, \ldots, t_n$ to a new root.



- $\delta_\bullet = f(x_0)$, if $t = B_+(t_1, \ldots, t_n)$,
  $\delta_t = f^{(n)}(x_0)\delta_{t_1} \ldots \delta_{t_n}$.
- What about

$$x(s) = x_0 + \sum_t \frac{s^{|t|}}{\sigma(t)} \varphi(t) \delta_t$$

Butcher's group

- What about

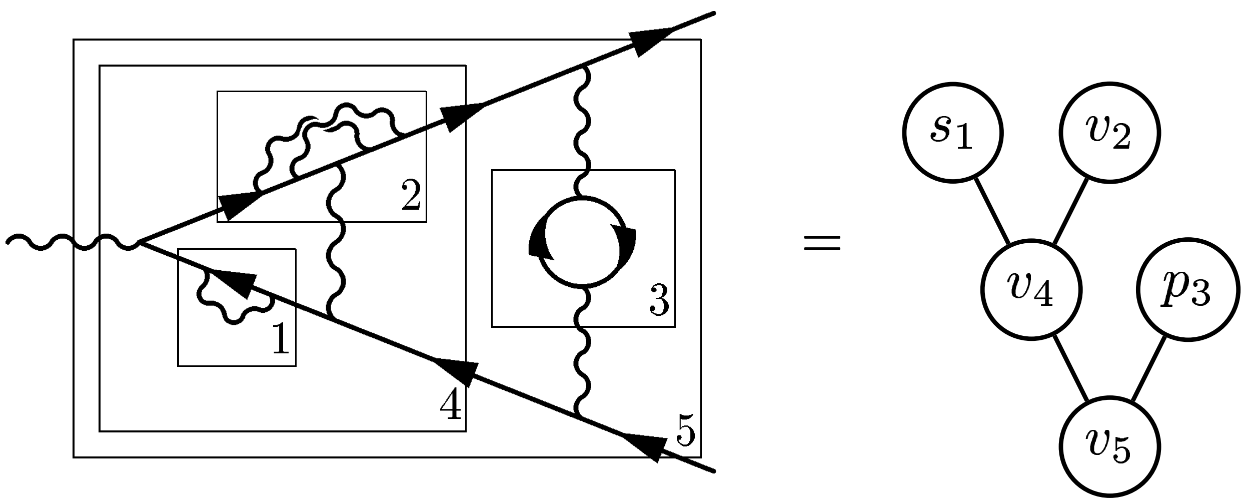
$$x(s) \;=\; x_0 + \sum_t \frac{s^{|t|}}{\sigma(t)} \varphi(t) \delta_t$$

- Runge-Kutta methods (Butcher, 1972)

$$\begin{aligned} x_i(s) &= x_0 + s \sum_{j=1}^{m} a_{ij} f(x_j(s)), \\ x(s) &= x_0 + s \sum_{j=1}^{m} b_j f(x_j(s)), \end{aligned}$$

- For any $\varphi(t)$, there is a Runge-Kutta method (A, b) giving $\varphi(t)$.
- Theorem: *The Runge-Kutta methods form a group.*
- Product: $(\varphi \star \varphi')(t) = \sum \varphi(t_{(1)}) \varphi'(t_{(2)})$.
  $(\varphi \star \varphi')(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}) = \varphi(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}) + \varphi'(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}) + \varphi(\bullet)\varphi'(\bullet)$.
- Unit: $\varepsilon(t)$.
- Inverse: $\varphi^{-1}(t) = \varphi(S(t))$.
- The Hopf algebra of Butcher's group: Arne Dür (1986).

Trees and renormalisation: quantum electrodynamics (Kreimer, Wulkenhaar, Krajewski)



- The big divergent vertex diagram 5 contains a divergent vacuum polarization diagram 3 and a divergent vertex diagram 4
- The divergent vertex diagram 4 contains a divergent self-interaction diagram 1 and a divergent vertex diagram 2

Partial differential equations

- $Lg(\mathbf{r}) = F(g(\mathbf{r}))$

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) g(x,y) = g(x,y)^3.$$

- Tree solution

$$g(\mathbf{r}) \quad = \quad g_0(\mathbf{r}) + \sum_t \frac{1}{\sigma(t)} \varphi(t;\mathbf{r}),$$

- where

$$\varphi(\bullet;\mathbf{r}) \quad = \quad \int \mathrm{d}\mathbf{r}' G(\mathbf{r},\mathbf{r}') F(g_0(\mathbf{r}')),$$

$$\varphi(t;\mathbf{r}) \quad = \quad \int \mathrm{d}\mathbf{r}' G(\mathbf{r},\mathbf{r}') F^{(n)}(g_0(\mathbf{r}')) \\ \varphi(t_1;\mathbf{r}') \ldots \varphi(t_n;\mathbf{r}'),$$

  for $t = B_+(t_1, \ldots, t_n)$.

- g_0 is a solution of $Lg_0 = 0$.
- $G(\mathbf{r},\mathbf{r}')$ is the Green function of L:

$$LG(\mathbf{r},\mathbf{r}') \quad = \quad \delta(\mathbf{r}-\mathbf{r}').$$

Chen's iterated integrals (Kreimer)

- Definition of $\varphi(t;x)$

$$\begin{aligned}\varphi(\mathbf{1};x) &= 1,\\ \varphi(\bullet;x) &= \int_x^\infty f_1(y)\mathrm{d}y,\\ \varphi(t;x) &= \int_x^\infty f_n(y)\varphi(t_1;y)\dots\varphi(t_n;y)\mathrm{d}y\end{aligned}$$

  if $t = B_+(t_1,\dots,t_n)$.

- In QFT, each integral $\varphi(t;x)$ is divergent. How can we remove the subdivergences (of $\varphi(t_k;y)$) and the divergence of $\varphi(t;x)$?

- Product

$$\varphi(t_1\dots t_n;x) = \varphi(t_1;x)\dots\varphi(t_n;x).$$

- Finite part: $\Gamma_{ab}(\mathbf{1}) = 1$ and

$$\Gamma_{ab}(t) = \sum \varphi\big(S(t_{(1)});a\big)\varphi(t_{(2)};b).$$

- Renormalisation group property

$$\Gamma_{ac}(t) = \sum \Gamma_{ab}(t_{(1)})\Gamma_{bc}(t_{(2)}).$$

Example

- Definition

$$
\begin{aligned}
\varphi(\bullet; x) &= \int_x^\infty f(y)\mathrm{d}y, \\
\varphi\left(\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}; x\right) &= \int_x^\infty f(y)\mathrm{d}y \int_y^\infty f(u)\mathrm{d}u.
\end{aligned}
$$

- Renormalisation

$$
\begin{aligned}
\Gamma_{ab}(t) &= \sum \varphi\big(S({\color{red}t_{(1)}}); a\big)\varphi(t_{(2)}; b), \\
\Delta \begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix} &= {\color{red}\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}} \otimes \mathbf{1} + \mathbf{1} \otimes \begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix} + {\color{red}\bullet} \otimes \bullet, \\
\Gamma_{ab}\left(\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}\right) &= \varphi\left(S\left({\color{red}\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}}\right); a\right) + \varphi\left(\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}; b\right) + \varphi(S({\color{red}\bullet}); a)\varphi(\bullet; b).
\end{aligned}
$$

- Antipode

$$
\begin{aligned}
S({\color{red}\bullet}) &= -{\color{red}\bullet}, \\
S\left({\color{red}\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}}\right) &= -{\color{red}\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}} + {\color{red}\bullet\bullet}.
\end{aligned}
$$

- Renormalisation

$$
\begin{aligned}
\Gamma_{ab}\left(\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}\right) &= -\varphi\left({\color{red}\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}}; a\right) + \varphi({\color{red}\bullet}; a)\varphi({\color{red}\bullet}; a) + \varphi\left(\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}; b\right) \\
&\quad -\varphi({\color{red}\bullet}; a)\varphi(\bullet; b) \\
&= \int_b^a f(y)\mathrm{d}y \int_y^a f(u)\mathrm{d}u.
\end{aligned}
$$

The Connes-Moscovici Hopf algebra

- Index theorem for the noncommutative geometry of foliations
- Hopf algebra generated by $\mathbf{1}, X, Y, \delta_n$.
- Relations

$$\begin{aligned} [Y, X] &= X, \quad [X, \delta_n] = \delta_{n+1}, \\ [Y, \delta_n] &= n\delta_n, \quad [\delta_n, \delta_m] = 0. \end{aligned}$$

- Coproduct

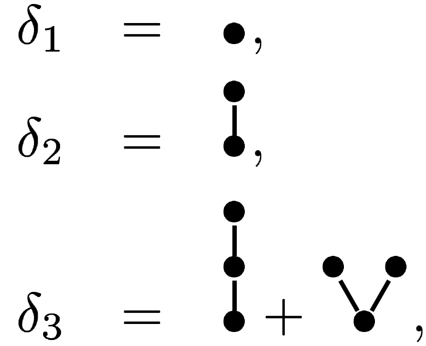
$$\begin{aligned} \Delta Y &= Y \otimes \mathbf{1} + \mathbf{1} \otimes Y, \\ \Delta X &= X \otimes \mathbf{1} + \mathbf{1} \otimes X + \delta_1 \otimes Y, \\ \Delta \delta_1 &= \delta_1 \otimes \mathbf{1} + \mathbf{1} \otimes \delta_1. \end{aligned}$$

- Recurrence

$$\begin{aligned} \Delta \delta_2 &= \Delta([X, \delta_1]) = \Delta(X\delta_1) - \Delta(\delta_1 X), \\ &= \delta_2 \otimes \mathbf{1} + \mathbf{1} \otimes \delta_2 + \delta_1 \otimes \delta_1, \\ \Delta \delta_3 &= \delta_3 \otimes \mathbf{1} + \mathbf{1} \otimes \delta_3 + \delta_2 \otimes \delta_1 + 3\delta_1 \otimes \delta_2 \\ &\quad + \delta_1^2 \otimes \delta_1 \end{aligned}$$

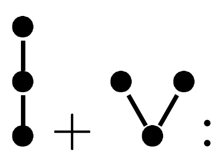
The connection

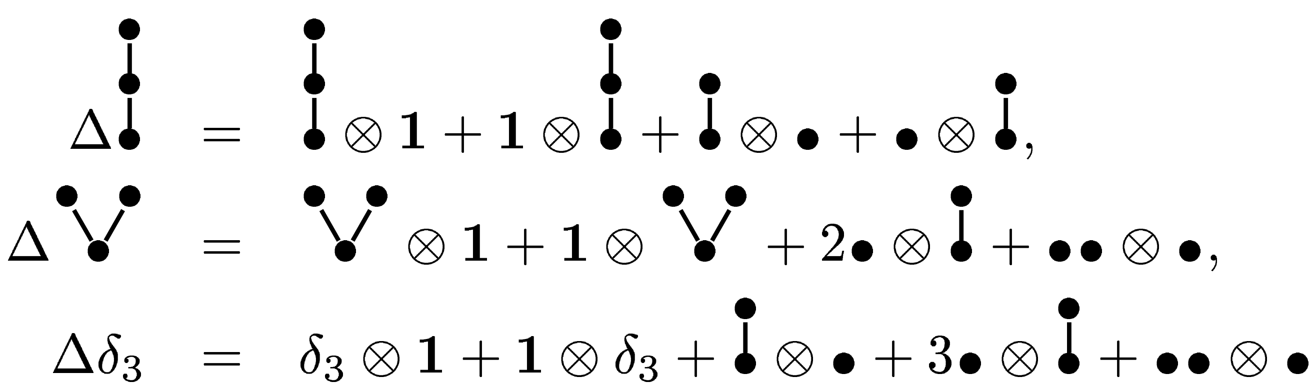
- Butcher's Hopf algebra is a refinement of Connes and Moscovici's



$$\delta_n = \sum_{|t|=n} t,$$

$$\Delta_{CM}\delta_n = \sum_{|t|=n} \Delta_B t.$$

- Example $\delta_3 =$  :



$$= \delta_3 \otimes \mathbf{1} + \mathbf{1} \otimes \delta_3 + \delta_2 \otimes \delta_1 + 3\delta_1 \otimes \delta_2$$

$$+\delta_1^2 \otimes \delta_1.$$

Composition of series

- Factorial series

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} f_n \frac{x^n}{n!}, \quad g(x) = \sum_{n=1}^{\infty} g_n \frac{x^n}{n!}, \\
f(g(x)) &= g_1 f_1 x + (g_2 f_1 + g_1^2 f_2) \frac{x^2}{2!} \\
&\quad + (g_3 f_1 + 3 g_1 g_2 f_2 + g_1^3 f_3) \frac{x^3}{3!} + \dots
\end{aligned}
$$

- Linear forms u_n, $\langle u_n, f \rangle = f_n$.
- Coproduct of u_n

$$
\begin{aligned}
\langle u_n, f \circ g \rangle &= \langle \Delta u_n, g \otimes f \rangle, \\
&= \sum \langle u_{n(1)}, g \rangle \langle u_{n(2)}, f \rangle.
\end{aligned}
$$

- Examples

$$
\begin{aligned}
\Delta u_1 &= u_1 \otimes u_1, \\
\Delta u_2 &= u_2 \otimes u_1 + u_1^2 \otimes u_2, \\
\Delta u_3 &= u_3 \otimes u_1 + 3 u_1 u_2 \otimes u_2 + u_1^3 \otimes u_3.
\end{aligned}
$$

Relation with Connes-Moscovici

- Generating function for Faà di Bruno

$$U(x) \quad = \quad x + \sum_{n=2}^{\infty} u_n \frac{x^n}{n!}.$$

- Generating function for Connes-Moscovici

$$D(x) \quad = \quad \sum_{n=1}^{\infty} \delta_n \frac{x^n}{n!}.$$

- Relation

$$D(x) \quad = \quad \log(U'(x)).$$

- Examples

$$\begin{aligned}
\delta_1 &= u_2, \\
\delta_2 &= u_3 - u_2^2, \\
\delta_3 &= u_4 - 3u_3u_2 + 2u_2^3, \\
\Delta_{CM}\delta_3 &= \Delta_F(u_4 - 3u_3u_2 + 2u_2^3).
\end{aligned}$$

Facts and open questions

- The relation between the compositions of functions and renormalisation is the renormalisation group.
- The relation between the compositions of Runge-Kutta methods and of the composition of functions was given by Hairer and Wanner.
- What is the relation between Runge-Kutta methods and renormalisation? A more general renormalisation group?
- Is it possible to connect Butcher's solution of differential equations with the renormalisation group method for differential equations (T. Kunihiro, N. Goldenfeld, Y. Oono, J. Bricmont, A. Kupiainen)?
- What is the use of the noncommutative analogues of Butcher's algebra?

Noncommutative extensions

- Dendriform algebras. (J.-L. Loday and M.O. Ronco, 1998)
- Operads (J. Moerdijk, M. Livernet, F. Chapoton 2001, P. van der Laan, 1999, 2001)
- Composition of functions of noncommutative variables (Ch. B, A. Frabetti, 2000)
- Noncommutative Butcher algebra (R. Holtkamp, L. Foissy, 2001)
- All these Hopf algebras are equivalent (R. Holtkamp, 2001, L. Foissy, 2001, P. Palacios, 2002)

The binomial Hopf algebra

- Generated by $\partial = \frac{d}{dx}$
- $\mathcal{A} =$ polynomials in the variable ∂
- Product $\partial^m \partial^n = \partial^{m+n}$
- Coproduct

$$\Delta \partial^n = \sum_{k=0}^{n} \binom{n}{k} \partial^k \otimes \partial^{n-k}.$$

- A counit $\varepsilon(\partial^n) = \delta_{n,0}$
- An antipode $S(\partial^n) = (-1)^n \partial^n$

The Faà di Bruno algebra

- $\mathcal{A}$ is the polynomial algebra in $u_1, u_2, \ldots$.
- Coproduct

$$\Delta u_n = \sum_{k=1}^{n} \sum_{\alpha} \frac{n!(u_1)^{\alpha_1} \ldots (u_n)^{\alpha_n}}{\alpha_1! \ldots \alpha_n! (1!)^{\alpha_1} \ldots (n!)^{\alpha_n}} \otimes u_k,$$

  where the sum is over the n-tuples of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = k$.

- Hopf algebra $u_1 = \mathbf{1}$

$$f(x) = x + \sum_{n=2}^{\infty} f_n \frac{x^n}{n!}.$$

- Counit $\varepsilon(\mathbf{1}) = 1$, $\varepsilon(u_n) = 0$ for $n > 1$.
- Antipode: inversion of series

$$\langle S(u_n), f \rangle = \langle u_n, f^{-1} \rangle.$$

Relation with Connes-Moscovici

- General form of δ_n

$$\delta_n \quad = \quad \sum_{k=0}^{n-1}(-1)^k k! \sum_{\alpha} \frac{n!(u_2)^{\alpha_1} \ldots (u_{n+1})^{\alpha_n}}{\alpha_1! \ldots \alpha_n!(1!)^{\alpha_1} \ldots (n!)^{\alpha_n}}.$$

- General form of u_n

$$u_{n+1} \quad = \quad \sum_{\alpha} \frac{n!(\delta_1)^{\alpha_1} \ldots (\delta_n)^{\alpha_n}}{\alpha_1! \ldots \alpha_n!(1!)^{\alpha_1} \ldots (n!)^{\alpha_n}}.$$

- Relation between trees and δ_n
- Relation between δ_n and Faà di Bruno
- Composition of series over trees?