Solving parabolic Wick-stochastic boundary value problems using a finite element method

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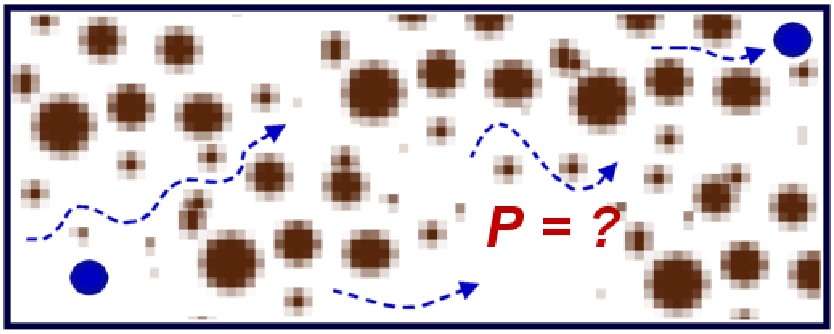
Overview

1. Introduction
2. Problem formulation
3. Numerical approximation
4. Example

Example

Flow in a porous medium ..



$$-\nabla(K(x)\cdot\nabla p(x)) = f(x), \\ p|_{\partial\mathcal{D}}(x) = 0 \tag{1}$$

.. with random permeability [Holden et al.]

Model: $K =$ "exponential" of white noise

Remarks

- Singular solution and noise.
- Multiplicative singular noise.
- What does $K(x)\cdot\nabla p(x)$ mean?

Generalized stochastic variables

Example Kondratiev spaces

$$(\mathcal{S})^1 \subset L^2(\Omega, \mathcal{B}, d\mu) \subset (\mathcal{S})^{-1}$$

Assume: Solution p is a *generalized* stochastic process

$$x \;\mapsto\; p(x,\cdot) \;\in\; (\mathcal{S})^{-1}$$

Advantages

- Stochastic analog to Schwary distributions

$$\mathcal{S}(\mathrm{R}^d) \subset L^2(\mathrm{R}^d, \mathcal{F}, dx) \subset \mathcal{S}'(\mathrm{R}^d)$$

- May define a product (Wick product $\diamond$)
- Powerful tools for solution
  (Hermite transform, Charecterization theorem)
- Allows for strong solutions (e.g., $C^{1,2}\,(0, T; (\mathcal{S})^{-1}))$

Problem formulation

$$\begin{aligned} D_t u + L^{\diamond} u &= f, \quad \text{on} \quad (0,T) \times \mathcal{D} \times \Omega \\ u|_{\partial\mathcal{D}} &= 0, \\ u|_{t=0} &= u_0, \end{aligned} \tag{2}$$

were

$$L^{\diamond} u = -D_i(a^{ij} \diamond D_j u) + b^i \diamond D_i u + c \diamond u \tag{3}$$

- a^{ij}, b^i, c and f are generalized stochastic processes.
- Products understood as Wick products ($\diamond$).
- $\mathcal{D} \subset \mathrm{R}^d$ open and bounded, $\partial\mathcal{D}$ smooth, and $T < \infty$.

Some notation and results

Probability space: $(\mathcal{S}'(\mathrm{R}^d), \mathcal{B}, \mu)$

$$\int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|_0^2} \qquad (\phi \in \mathcal{S}(\mathrm{R}^d))$$

Hermite polynomials:

$$H_\alpha(\omega) := \prod_{j=1}^{\infty} h_{\alpha_j}(\langle \omega, \eta_j \rangle),$$

$$(\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{I} := (\mathrm{N}_0)_c^\infty)$$

Chaos expansion: If $f \in L^2(\mu)$ then

$$f = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha \quad \text{and} \quad \|f\|_{L^2(\mu)}^2 = \sum_{\alpha \in \mathcal{I}} f_\alpha^2 \alpha!$$

Wick product: $f = \sum_\alpha f_\alpha H_\alpha$, $g = \sum_\beta g_\beta H_\beta$,

$$f \diamond g = \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \beta = \gamma} f_\alpha g_\beta \; H_\gamma$$

A class of spaces

Definition $k \in \mathrm{R}$, $\rho \in [-1,1]$, X separable Hilbert space

$$(\mathcal{S})^{\rho,k,X} := \{f = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha : f_\alpha \in X, \quad \|f\|_{\rho,k,X} < \infty\}$$

where

$$\|f\|^2_{\rho,k,X} := \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_X (\alpha!)^{1+\rho} (2\mathrm{N})^{k\alpha}, \quad (2\mathrm{N})^{k\alpha} := \prod_{j=1}^{\infty} (2j)^{k\alpha_j}$$

[Våge (1994)]

Remark

$$(\mathcal{S})^1 = \bigcap_k (\mathcal{S})^{1,k,\mathrm{R}} \quad \text{(projective limit)}$$

$$(\mathcal{S})^{-1} = \bigcup_k (\mathcal{S})^{-1,k,\mathrm{R}} \quad \text{(inductive limit)}$$

Notation

$$(\mathcal{S})^{-1,k,m} := (\mathcal{S})^{-1,k,X} \quad \text{with} \quad X = H^m(\mathcal{D})$$

$$(\mathcal{S})^{-1,k,1}_0 := (\mathcal{S})^{-1,k,Y} \quad \text{with} \quad Y = H^1_0(\mathcal{D})$$

The Wick product

Recall $$f \diamond g = \sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta H_\gamma$$

A Banach space: For $l \in \mathrm{R}$,

$$\mathcal{F}_l := \{f(x) = \sum_{\alpha \in \mathcal{I}} f_\alpha(x) H_\alpha \ : f_\alpha \text{ measurable}, \\ \|f\|_{l,*} < \infty\}$$

where $$\|f\|_{l,*} := \sup_{x \in \mathcal{D}} (\sum_\alpha |f_\alpha(x)| (2\mathrm{N})^{l\alpha}).$$

Proposition 1 [Våge (1994)]
Given $k, l \in \mathrm{R}, k \leq 2l$. If $f \in \mathcal{F}_l$ it holds

$$\|f \diamond g\|_{-1,k,0} \leq \|f\|_{l,*} \|g\|_{-1,k,0},$$

for all $g \in H$.

Note: For $f \in \mathcal{F}_l$, the map $g \mapsto f \diamond g$ is linear and continuous on $(\mathcal{S})^{-1,k,0}$ $(k \leq 2l)$

The variational formulation of (2)

Recall (2)

$$D_t u + L^\diamond u = f, \quad \text{on} \quad (0,T) \times \mathcal{D} \times \Omega$$
$$u|_{\partial\mathcal{D}} = 0, \quad u|_{t=0} = u_0,$$

Notation For $k \in \mathrm{R}$,

$$V := (\mathcal{S})_0^{-1,k,1}, \text{ and } \|\cdot\| := \|\cdot\|_V$$
$$H := (\mathcal{S})^{-1,k,0}, \text{ and } (\cdot,\cdot) := (\cdot,\cdot)_H$$

Note: $V \hookrightarrow H \hookrightarrow V'$ (dense injection)

Idea: Take H-inner product with $v \in V$ in (2).

Assume

$$(\mathrm{A}_1) \quad a^{ij}, b^i, c \in F_l(\mathcal{D}), \quad (k \le 2l)$$
$$(\mathrm{A}_2) \quad f \in L^2(0,T;V'),$$

and let $u \in W(0,T;V)$ where

$$W(0,T;V) := \{g \in L^2(0,T;V) : D_t g \in L^2(0,T;V')\}.$$

Variational formulation

Find $u \in W(0,T;V)$ s.t.

$$\frac{d}{dt}(u(\cdot),v) + \mathcal{A}(u(\cdot),v) = (f(\cdot),v), \qquad (4)$$
$$u(0) = u_0$$
$$(v \in V)$$

where

$$\mathcal{A}(u,v) = (a^{ij}\diamond D_j u, D_i v) + (b^i \diamond D_i u, v) + (c \diamond u, v).$$

Remarks

- $u(0)$ has sense $\quad (W(0,T;V) \subset C^0(0,T;H)\ )$
- (A) $\Rightarrow$ $\mathcal{A}$ is a continuous bilinear form on V.
- Equation understood in $\mathcal{D}'(0,T)$.

Existence and uniqueness

Define: Bilinear form on $H_0^1(\mathcal{D})$

$$\mathcal{B}_\beta(g,h) := (a_\beta^{ij} D_j g, D_i h)_0 + (b_\beta^i D_i g, h)_0 + (c_\beta g, h)_0$$

$$(g, h \in H_0^1(\mathcal{D}))$$

Theorem 2 [Theting (2001)]
Assume (A) and

(B) There exists $\theta_0 > 0$ s.t.

$$\mathcal{B}_0(g,g) \geq \theta_0 \, \|g\|_1^2 \qquad (g, h \in H_0^1(\mathcal{D}))$$

Then for k small enough, there exists a unique solution to the variational problem (4).

Example

Find $p \in W(0,T;V)$ s.t. $p(0) = p_0$,

$$\frac{d}{dt}(p(\cdot), v) + (K \diamond \nabla p(\cdot), \nabla v) = (f(\cdot), v), \qquad (v \in V) \tag{5}$$

Easy to show:

$$K(\cdot) := \exp^\diamond(W_\cdot) := \sum_{n=0}^{\infty} \frac{1}{n!} W_\cdot^{\diamond n} \in \mathcal{F}_l \text{ for } l < -1 \Rightarrow (A) \text{ ok}$$

$$\mathcal{B}_0(g,h) = (\nabla g, \nabla h)_0 \qquad \Rightarrow (B) \text{ ok}$$

Numerical approximation

Idea 1 Approximate V by a finite-dimensional subspace V_m.

Semi-discrete problem

$$\begin{aligned} &\text{Find } u_m \in W(0,T;V_m) \text{ s.t.} \\ &(D_t u_m(t), v) + \mathcal{A}(u_m(t), v) = (f(t), v), \\ &\qquad u_m(0) = P_m u_0 \\ &\qquad\qquad (v \in V_m) \end{aligned} \tag{6}$$

(P_m is a projection onto V_m).

Idea 2 Approximate $D_t u_m$ using a finite difference method (e.g., Backward Euler).

Fully discrete problem

$$\begin{aligned} (\frac{1}{\Delta t}(U^{n+1} - U^n), v) + \mathcal{A}(U^{n+1}, v) &= (f(t_{n+1}), v), \\ U^0 &= P_m u_0 \\ &\qquad (v \in V_m) \end{aligned} \tag{7}$$

($U^n \approx u_m(t_n)$, $t_n := n \cdot \Delta t$).

Construction of V_m

(a) Cut off chaos expansion

$$\mathcal{I}_{N,K} := \{0\} \cup \bigcup_{n=1}^{N} \bigcup_{k=1}^{K} \{\alpha \in \mathbb{N}_0^k \ : \ |\alpha| = n, \alpha_k \neq 0\}$$

(b) Approximate $H_0^1(\mathcal{D})$ using a FE-space

$$X_M := \{v \in C^0(\bar{\mathcal{D}}) \ : \ v|_K \text{ linear}, \ v|_{\partial\mathcal{D}} = 0\}$$

$$(K \in \mathcal{T}_h, M := \dim(X_M) < \infty, h := \max(\text{diam } (K)))$$

Define

$$V_m := \{f = \sum_{\alpha \in \mathcal{I}_{N,K}} f_\alpha^M H_\alpha \ : \ \|f\| < \infty\}$$

$$m = \min(M, N, K)$$

Lemma 3 [Theting (2001)]

$\{V_m : (N, K, M) \in \mathbb{N}^3\}$ is a Galerkin approximation of V

Error in approximation

Theorem 4 [Theting (2001)]

Assume

(C$_1$) $k \leq 0, \quad |k| = q + r, \; q \geq 0, \; r > 1.55$
(C$_2$) $u \in C^1(0, T; V)$
(C$_3$) $D_t u \in L^2(0, T; V \cap (\mathcal{S})^{-1,-|k|,2})$
(C$_4$) $D_t f \in L^2(0, T, V')$

and $U^0 = u_m(0) = Ru_0$ ($R : V \to V_m$ elliptic projection).

Then

$$\|u(t_n) - U^n\|^2_{-1,-|k|,0}$$
$$\leq c(K^{1-r} + 2^{-rN})\left[\|u(t_n)\|^2_{-1,-q,1} + \int_0^{t_n} \|D_s u(s)\|^2_{-1,-q,1}\, ds\right]$$
$$+ ch^2 \left[\|u(t_n)\|^2_{-1,-|k|,2} + \int_0^{t_n} \|D_s u(s)\|^2_{-1,-|k|,2}\, ds\right]$$
$$+ c\Delta t^2 \left[\int_0^{t_n} (\|D_s u(s)\|^2_V + \|D_s f(s)\|^2_{V'}) ds\right]$$

Example

The time-dependent pressure equation

$$\text{Find } p \in W(0,T;V) \text{ s.t. } p(0) = p_0\,, \qquad \frac{d}{dt}(p(\cdot),v) + (K\diamond\nabla p(\cdot),\nabla v) = (f,v), \quad (v \in V) \tag{8}$$

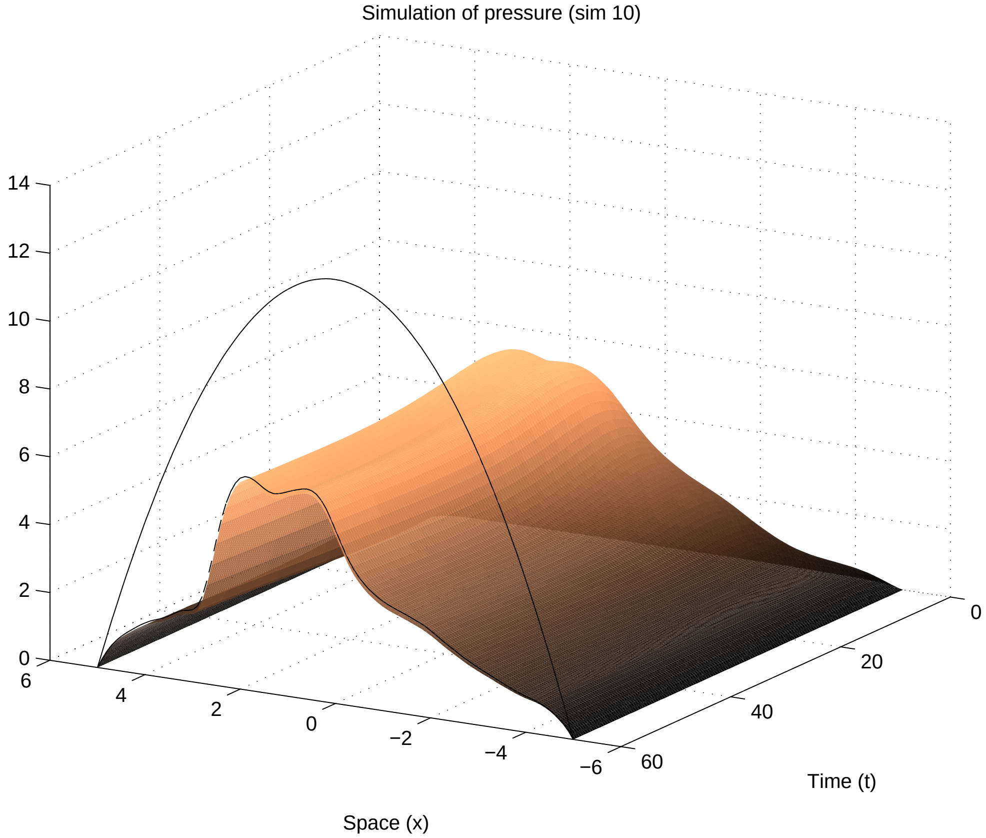


Figure 1 *Typical time development of a simulation.*

$$f := 1,\ K(x) := \exp^{\diamond}(W_x),$$
$$\mathcal{D} = [-5,5],\ (N,K,M) = (3,15,100),\ \Delta t = 0.2.$$

The Pressure equation

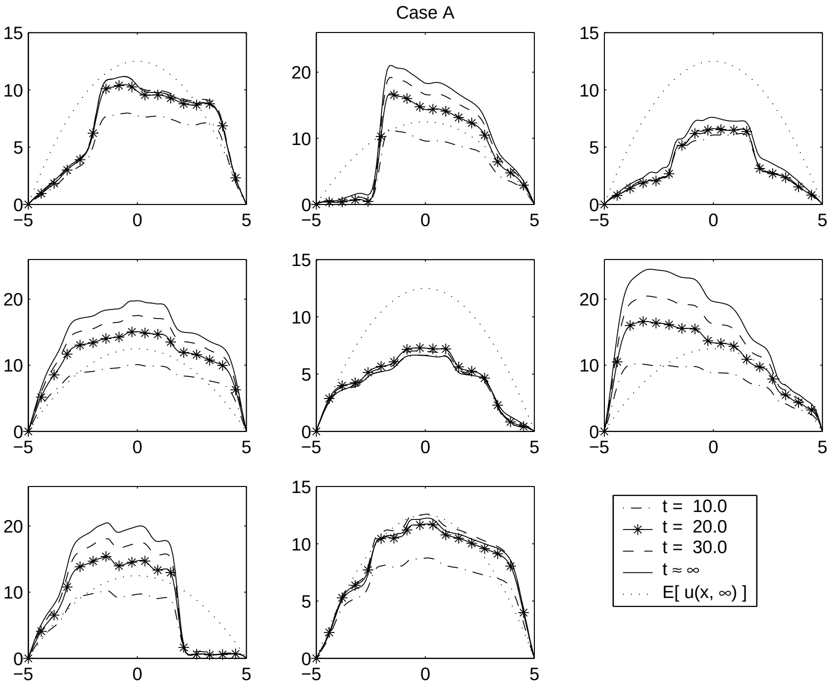


Figure 2 *Eight different simulations of the pressure.*

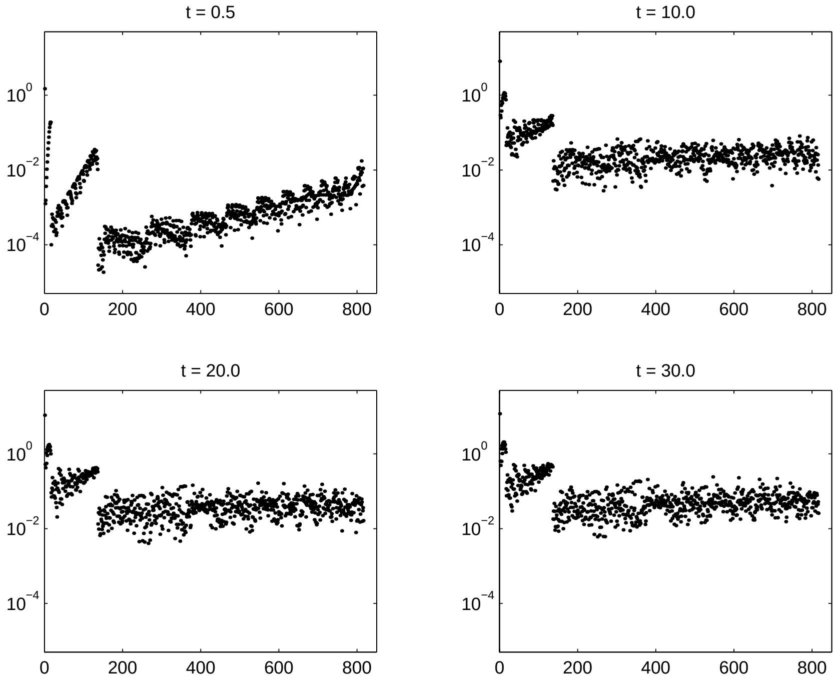


Figure 3 *The sup-norm of the chaos coefficients.*

Example

The Poisson equation with white noise

$$\Delta u(x) = W_x \quad (\text{ on } \mathcal{D}), \qquad u|_{\partial\mathcal{D}} = 0$$

- Stationary problem.
- Set $\mathcal{D} = [-5, 5]$ and $(N, K, M) = (1, 1000, 800)$.

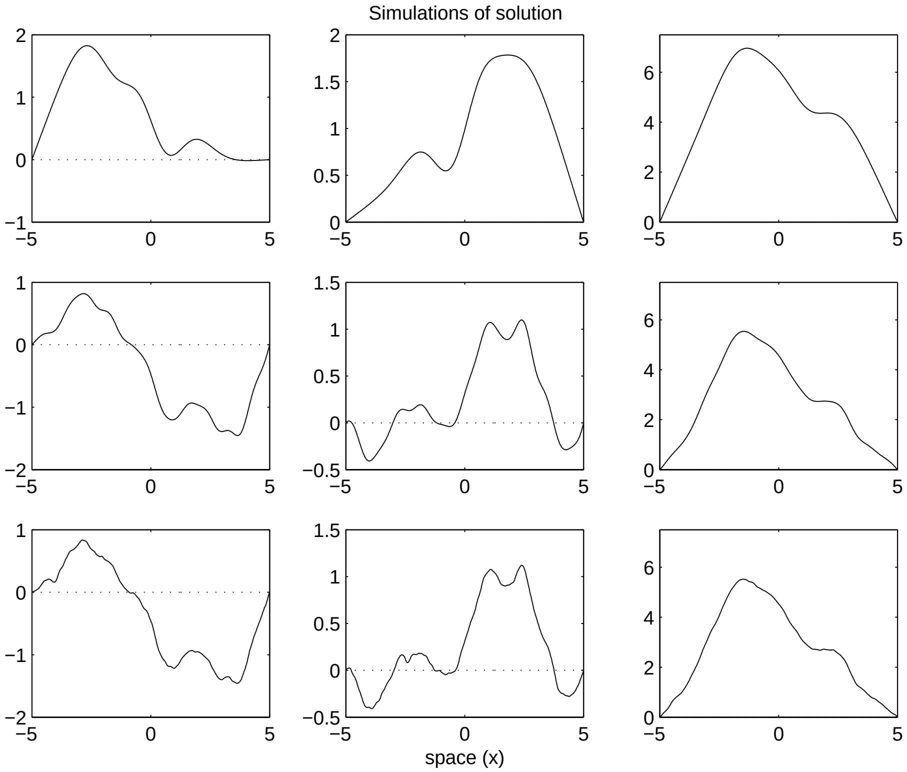


Figure 4 *Three different simulations of solution. Each row corresponds to* $K = 6$*,* 35 *and* 1000*, respectively.*

The Poisson equation

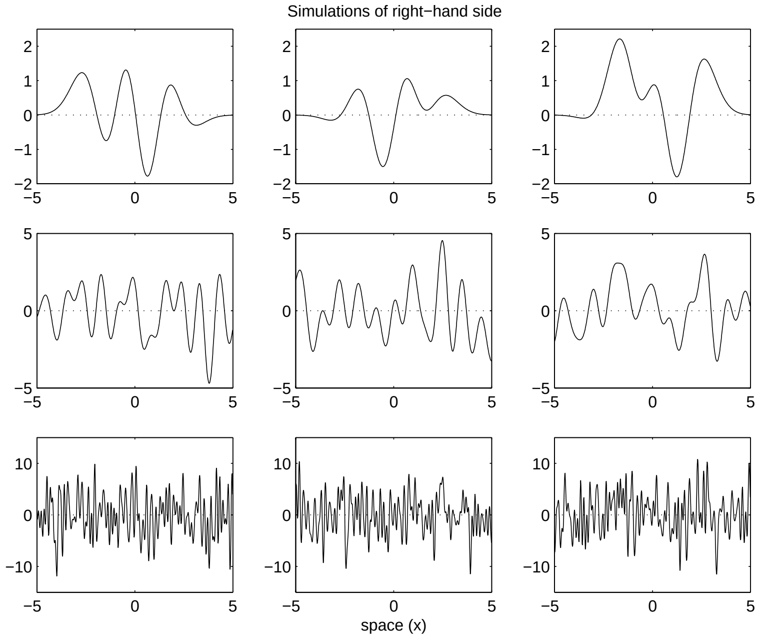


Figure 5 *The corresponding simulations of right-hand side. Each row corresponds to* $K = 6$, 35 *and* 1000, *respectively.*

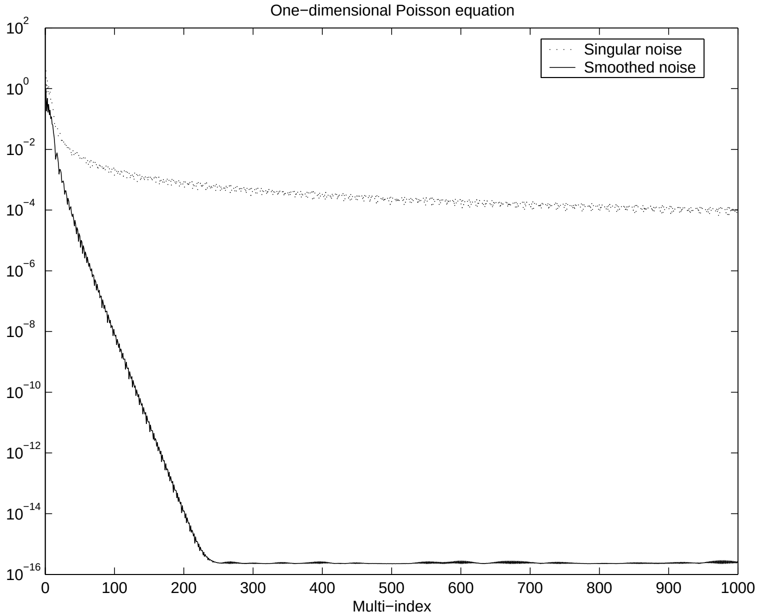


Figure 6 *Sup-norm of chaos coefficients when right-hand side is singular- and smoother white noise*

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Notation

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$$(2\mathrm{N})^{k\alpha} := \prod_{j=1}^{\infty} (2j)^{k\alpha_j}$$

$$V := (\mathcal{S})_0^{-1,k,1}, \quad \|\cdot\| = \|\cdot\|_{-1,k,1}$$
$$H := (\mathcal{S})^{-1,k,0}, \quad (\cdot,\cdot) = (\cdot,\cdot)_{-1,k,0}$$

$$\mathcal{A}(u,v) = (a^{ij}\diamond D_j u, D_i v) + (b^i \diamond D_i u, v) + (c\diamond u, v).$$

$$\mathcal{B}_\beta(g,h) := (a^{ij}_\beta D_j g, D_i h)_0 + (b^i_\beta D_i g, h)_0 + (c_\beta g, h)_0$$