Lectures on Weak Convergence Methods and Recent Mathematical Theory for the Compressible Navier-Stokes Equations

A basic problem in nonlinear partial differential equations (PDEs) goes roughly speaking a follows: We wish to solve a nonlinear PDE (or system of such), which is here written abstractly as

$$F[v] = f, \tag{0.0.1}$$

where $F[\cdot]$ is a nonlinear operator, f is a given function, and v is the sought unknown function.

To establish existence of a solution we shall solve approximate problems (for which existence is easy)

$$F_n[v_n] = f_n, \qquad n = 1, 2, \dots,$$

where $F_n[\cdot]$ is a (nonlinear) operator which is in some sense close to F, f_n is close to f, and v_n is the (approximate) solution. The approximate problems can for example be related to a numerical discretization of the original problem. Here is the

> **basic problem:** Show that the sequence $\{v_n\}_{n\geq 1}$ of approximate solutions converges to a solution of the original problem (0.0.1).

This is a highly non-trivial problem due to the nonlinearity of the governing PDE! In many (if not most) cases, we will be able to prove a priori bounds on $\{v_n\}_{n\geq 1}$ that only imply weak – *but not strong* – convergence along a subsequence of $\{v_n\}_{n\geq 1}$ to some limit function v (for example in suitable L^p and/or Sobolev spaces). In this case we write

$$v_n \rightharpoonup v \quad \text{as } n \uparrow \infty.$$

However, it is not clear at all that this weak convergence implies (in any reasonable sense)

$$F_n[v_n] \to F[v] \quad \text{as } n \uparrow \infty,$$

since we know from functional analysis that the weak topology is not compatible with a nonlinear structure (unless convexity is present, which is not the case in many instances).

Nonlinearity destroys weak convergence! However, what saves the day in many nonlinear PDEs is that the particular structure of the nonlinearity, which is unfortunately often hidden(!), provide enough information to justify passing to the limit.

These lectures will attempt to survey some advances in weak convergence methods for nonlinear PDEs and in particular the multidimensional compressible isentropic Navier-Stokes equations. This system of nonlinear PDEs reads

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho u) = 0, \\ &\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla p = \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u, \end{aligned}$$

where the sought unknowns are the density $\varrho \geq 0$ and the velocity $u = (u_1, \ldots, u_N)^\top$ of the fluid. Moreover, p is the pressure, which is assumed to be given nonlinear function of the density (typically $p(\varrho) = a\varrho^\gamma$ for some constants $a > 0$ and $\gamma > 1$), and λ, μ are the Lamé viscosity (constant) coefficients of the fluid.

Recent spectacular development in the mathematical theory for compressible Navier-Stokes equations provide some new compactness machinery by which weak limits become strong. This lecture series will report on some of this development. It is partly based on the book [P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 2*, Oxford Univ. Press, New York, 1998; MR 99m:76001], which contains the first global-in-time existence result for weak solutions of the compressible isentropic Navier-Stokes equations. Some extensions of Lions' result due to E. Feireisl [99-03] and collaborators will also be discussed.

While the global-in-time existence of weak solutions to the compressible Navier-Stokes equations have remained open until recently, one should recall (to gain a little bit of perspective) that the comparable global-in-time existence result for the *incompressible* Navier-Stokes equations is classical and goes back to J. Leray's fundamental work in the 1930s.

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