## Rejection sampling

We discuss a general approach to generate samples from some target distribution with density $f(x)$, called rejection sampling, without actually sampling from $f(x)$.

### Rejection sampling

The goal is to effectively simulate a random number $X \sim f(x)$ using two independent random numbers

- $U \sim \mathrm{U}(0,1)$ and
- $X \sim g(x)$,

where $g(x)$ is called proposal density and can be chosen arbitrarily under the assumption that there exists an $c \geq 1$ with

$$f(x) \leq c \cdot g(x) \quad \text{for all } x \in \mathbb{R}\,.$$

## Rejection sampling - Algorithm

Let $f(x)$ denote the target density.

1. Generate $x \sim g(x)$
2. Compute $\alpha = \frac{1}{c} \cdot \frac{f(x)}{g(x)}$.
3. Generate $u \sim \mathcal{U}(0,1)$.
4. If $u \leq \alpha$ return $x$ (acceptance step).
5. Otherwise go back to (1) (rejection step).

Note $\alpha \in [0,1]$ and $\alpha$ is called acceptance probability.

Claim: The returned $x$ is distributed according to $f(x)$.

## Proof

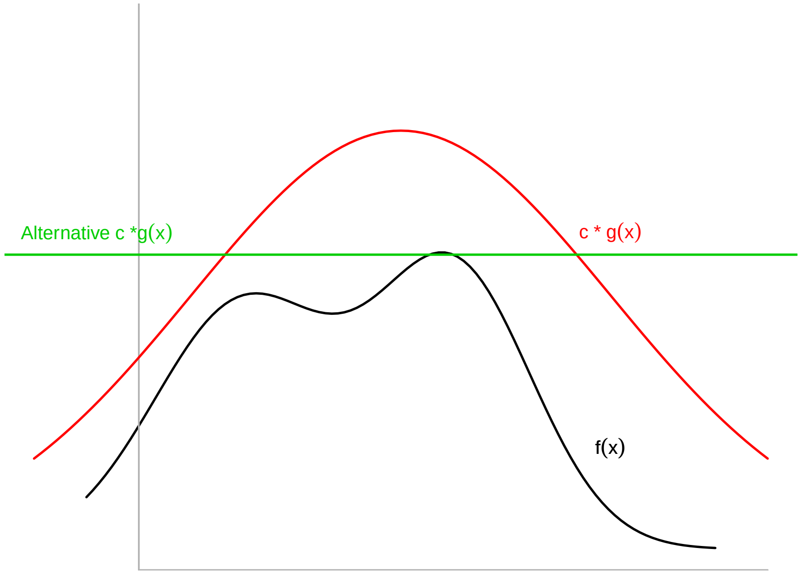
## Rejection sampling

- We want $x \sim f(x)$ (density).
- We know how to generate realisations from a density $g(x)$
- We know a a value $c > 1$, so that $\frac{f(x)}{g(x)} \leq c$ for all $x$ where $f(x) > 0$.

Algorithm:

finished $= 0$
while (finished $= 0$)
  generate $x \sim g(x)$
  compute $\alpha = \frac{1}{c} \cdot \frac{f(x)}{g(x)}$
  generate $u \sim U[0,1]$
  if $u \leq \alpha$ set finished $= 1$
return $x$

## Rejection sampling




## Rejection sampling

The overall acceptance probability is

$$P(c \cdot U \cdot g(x) \leq f(x)) = \int_{-\infty}^{\infty} \frac{f(x)}{c \cdot g(x)} g(x)\, dx = \int_{-\infty}^{\infty} \frac{f(x)}{c}\, dx = c^{-1}.$$

The single trials are independent, so the number of trials up to the first success is geometrically distributed with parameter $1/c$. The expected number of trials up to the first success is therefore $c$.

Problem:

In high-dimensional spaces $c$ is generally large so that many samples will get rejected.

## Continuation: Standard Cauchy

How can we sample from the semi-unit circle?

## Rejection sampling - Acceptance probability

Note: For $c$ to be small, $g(x)$ must be similar to $f(x)$.

The art of rejection sampling is to find a g(x) that is similar to f(x) and which we know how to sample from.

Issues: $c$ is generally large in high-dimensional spaces, and since $\alpha = 1/c$, many samples will get rejected.