Lecture 11: Conjugate gamma-Poisson hierarchical model

Example from George et al. (1993) regarding the analysis of 10 power plants.

- y_i number of failures of pump i
- t_i length of operation time of pump i (in kilo hours)

Model:

$$y_i \mid \lambda_i \sim \text{Po}(\lambda_i t_i)$$

Conjugate prior for λ_i:

$$\lambda_i \mid \alpha, \beta \sim \text{G}(\alpha, \beta)$$

Hyper-prior on α and β:

$$\alpha \sim \text{Exp}(1.0) \qquad \beta \sim \text{G}(0.1, 10.0)$$

Conjugate gamma-Poisson hierarchical model (II)

The posterior of the 12 parameters $(\alpha, \beta, \lambda_1, \ldots, \lambda_{10})$ given $y_1, \ldots, y_{10}$ is proportional to

$$\begin{aligned}\pi(\alpha, \beta, \lambda_1, \ldots, \lambda_{10} \mid y_1, \ldots, y_{10}) &\propto \pi(\alpha)\pi(\beta) \prod_{i=1}^{10} [\pi(\lambda_i \mid \alpha, \beta)\pi(y_i \mid \lambda_i)] \\ &\propto e^{-\alpha}\beta^{0.1-1}e^{-10\beta} \left\{ \prod_{i=1}^{10} \exp(-\lambda_i t_i)\lambda_i^{y_i} \right\} \left\{ \prod_{i=1}^{10} \exp(-\beta\lambda_i)\lambda_i^{\alpha-1} \right\} \left[\frac{\beta^\alpha}{\Gamma(\alpha)} \right]^{10}.\end{aligned}$$

This posterior is not of closed form.

What are the full conditional distributions?

The GMRF approximation

Let $\boldsymbol{x}$ denote a GMRF with precision matrix $\boldsymbol{Q}$ and mean $\boldsymbol{\mu}$. Approximate

$$\pi(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y}) \propto \exp\left(-\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \sum_{i=1}^{n} \log \pi(y_i|x_i) \right)$$

by using a second-order Taylor expansion of $\log \pi(y_i|x_i)$ around $\boldsymbol{\mu}_0$, say.

Recall

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 = a + bx - \frac{1}{2}cx^2$$

with $b = f'(x_0) - f''(x_0)x_0$ and $c = -f''(x_0)$.

The GMRF approximation (II)

Thus,

$$\begin{aligned}\tilde{\pi}(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y}) &\propto \exp\left(-\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \sum_{i=1}^{n} (a_i + b_i x_i - 0.5c_i x_i^2) \right) \\ &\propto \exp\left(-\frac{1}{2}\boldsymbol{x}^T(\boldsymbol{Q} + \text{diag}(\boldsymbol{c}))\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x} \right)\end{aligned}$$

to get a Gaussian approximation with precision matrix $\boldsymbol{Q} + \text{diag}(\boldsymbol{c})$ and mean given by the solution of $(\boldsymbol{Q} + \text{diag}(\boldsymbol{c}))\boldsymbol{\mu} = \boldsymbol{b}$. The canonical parameterization is

$$\mathcal{N}_C(\mathbf{b}, \boldsymbol{Q} + \text{diag}(\boldsymbol{c}))$$

which corresponds to

$$\mathcal{N}((\boldsymbol{Q} + \text{diag}(\boldsymbol{c}))^{-1}\mathbf{b}, (\boldsymbol{Q} + \text{diag}(\boldsymbol{c}))^{-1}).$$

The GMRF approximation