Norwegian University of Science and Technology
Department of Mathematical Sciences

English

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TMA4295 Statistical inference

Saturday 17 December 2011 9:00–13:00

Permitted aids: Yellow A5 sheet with your own handwritten notes (stamped by the Department of Mathematical Sciences), *Tabeller og formler i statistikk* (Tapir forlag), *Matematisk formelsamling* (K. Rottmann), calculator HP 30s or Citizen SR-270X

Grades to be announced: 17 January 2012

In the grading each of the ten points counts equally.

You should demonstrate how you arrive at your answers (e.g. by including intermediate answers or by referring to theory or examples from the reading list).

Problem 1 Let X_1, X_2, …, X_n be independent random variables from a Poisson distribution having parameter $\lambda > 0$, that is, having probability mass function given by $f(x) = \lambda^x e^{-\lambda}/x!$, $x = 0, 1, 2, \ldots$. We want to examine some point estimators of $P(X = 0) = e^{-\lambda}$ based on X_1, X_2, …, X_n.

a) Find the maximum likelihood estimator $\hat{\theta}$ of $e^{-\lambda}$.

b) Show that the Cramér–Rao lower bound of the variance of an unbiased estimator of $e^{-\lambda}$ is $\lambda e^{-2\lambda}/n$.

Let $U_i = 1$ if $X_i = 0$ and $U_i = 0$ otherwise, $1 \le i \le n$. Another estimator of $e^{-\lambda}$, $\tilde{\theta} = \frac{1}{n}\sum_{i=1}^n U_i$, is proposed.

c) Find the asymptotic relative efficiency (ARE) of $\tilde{\theta}$ with respect to $\hat{\theta}$. Which estimator would you prefer?

d) Show that $\sum_{i=1}^n X_i$ is a complete sufficient statistic for λ. Find the conditional expected value $E(U_1 \mid \sum_{i=1}^n X_i = m)$, where m is a non-negative integer. Show that the unique best (uniform minimal variance) unbiased estimator of $e^{-\lambda}$ is $\theta^* = (1 - 1/n)^{n\bar{X}}$.

e) Find the exact variances of $\hat{\theta}$ and θ^*, and compute numerical values of the variances when $n = 20$ and $\lambda = 1$. (Hint: Use the moment generating function of $\bar{X}$.) Is $\hat{\theta}$ unbiased?

Problem 2 Let X_1, X_2, …, X_n be independent random variables from a distribution having probability density function given by $f(x) = 2\theta x e^{-\theta x^2}$ for $x > 0$ and $f(x) = 0$ otherwise, where $\theta > 0$ is a parameter.

a) Show that $2\theta X_1^2$ has the chi-squared distribution with 2 degrees of freedom.

b) Use the exact distribution of $2\theta \sum_{i=1}^n X_i^2$ to find a $1 - \alpha$ confidence interval for θ.

In the remaining, we want to test $H_0\colon \theta = \theta_0$ against the alternative $H_1\colon \theta \ne \theta_0$. Assume $n = 50$ and test size $\alpha = 0.05$.

c) Find a test based on the exact distribution under H_0 of $2\theta_0 \sum_{i=1}^n X_i^2$. For which $\theta_0 \sum X_i^2$ should H_0 be rejected?

d) Find a test based on an approximate distribution under H_0 of $\theta_0 \sum_{i=1}^n X_i^2/n$ according to the central limit theorem. For which $\theta_0 \sum X_i^2$ should H_0 be rejected?

e) Find a test based on an approximate distribution under H_0 of the likelihood ratio test statistic. For which $\theta_0 \sum X_i^2$ should H_0 be rejected? (You may use that $t - 50 \ln t = -143.7$ for $t = 37.39$ and for $t = 65.17$.)