

ON COMPLEXES OF FINITE COMPLETE INTERSECTION DIMENSION

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ABSTRACT. We study complexes of finite complete intersection dimension in the derived category of a local ring. Given such a complex of complexity c , we prove that the thick subcategory it generates contains complexes of all possible complexities at most c . In particular, we show that such a complex is virtually small, answering a question raised by Dwyer, Greenlees and Iyengar.

1. INTRODUCTION

Given a homologically finite complex over a local ring, the complexity of the complex measures the polynomial rate of growth in its minimal free resolution. The rings over which all such complexes have finite complexity are precisely the local complete intersections. If a complex has finite complete intersection dimension (see Section 3), then it behaves homologically as a complex over a complete intersection. In particular, complexes of finite complete intersection dimension have finite complexity.

The main result of this paper concerns the thick subcategory $\text{thick}_{D(A)}(M)$ generated by a nonzero homologically finite complex M of finite complete intersection dimension over a local ring A . We show that if the complexity of M is c , then $\text{thick}_{D(A)}(M)$ contains a nonzero complex of complexity t for every $0 \leq t \leq c$. In particular, we show that the thick subcategory generated by M contains a nonzero perfect complex, that is, a complex of finite projective dimension, thus answering a question raised in [8, Remarks 9.6].

2. NOTATION AND TERMINOLOGY

Let (A, \mathfrak{m}, k) be a local (meaning commutative Noetherian local) ring, and denote by $D(A)$ the derived category of (not necessarily finitely generated) A -modules. A complex

$$M: \cdots \rightarrow M_{n+1} \rightarrow M_n \rightarrow M_{n-1} \rightarrow \cdots$$

in $D(A)$ is *bounded below* if $M_n = 0$ for $n \ll 0$, and *bounded above* if $M_n = 0$ for $n \gg 0$. The complex is *bounded* if it is both bounded below and bounded above, and *finite* if it is bounded and degreewise finitely generated. The *homology* of M , denoted $H(M)$, is the complex with $H(M)_n = H_n(M)$, and with trivial differentials. When $H(M)$ is finite, then M is said to be *homologically finite*.

As shown for example in [10], when the complex M is homologically finite, it has a minimal free resolution. Thus, there exists a quasi-isomorphism $F \simeq M$, where F is a bounded below complex

$$\cdots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \cdots$$

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of finitely generated free modules, and where $\text{Im } d_n \subseteq \mathfrak{m} F_{n-1}$. This resolution is unique up to isomorphism, and so for each integer n the rank of the free module F_n is a well defined invariant of M . This is the n th *Betti number* $\beta_n(M)$ of M , and the corresponding generating function $\sum_{n=0}^{\infty} \beta_n(M) t^n$ is the Poincaré series $P(M, t)$ of M . The *complexity* of M , denoted $\text{cx } M$, is defined as

$$\text{cx } M \stackrel{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq a n^{t-1} \text{ for } n \gg 0\}.$$

This notion was introduced for modules over local rings by Avramov in [1] and [2], and adopted to complexes by Sather-Wagstaff in [12]. The complexity of a homologically finite complex is not necessarily finite. In fact, it follows from a theorem of Gulliksen (cf. [9]) that finiteness of the complexities of all homologically finite complexes in $D(A)$ is equivalent to A being a complete intersection ring. Gulliksen stated this result in terms of finitely generated modules, but the proof carries over to complexes. Note that a complex has complexity zero if and only if its Betti numbers eventually vanish. Thus, the complexes of complexity zero are precisely those of finite projective dimension, that is, the perfect complexes.

Suppose that our complex M is homologically finite, with a minimal free resolution F . Given a complex $N \in D(A)$, the complex $\text{Hom}_A(F, N)$ is denoted by $\mathbf{R} \text{Hom}_A(M, N)$. Up to quasi-isomorphism, this complex is well defined, hence so is the cohomology group

$$\text{Ext}_A^n(M, N) \stackrel{\text{def}}{=} \mathbf{H}_{-n}(\mathbf{R} \text{Hom}_A(M, N))$$

for every integer n . When $N = k$, it follows from the minimality of F that the differentials in the complex $\text{Hom}_A(F, k)$ are trivial. Consequently, we see that $\beta_n(M)$ equals $\dim_k \text{Ext}_A^n(M, k)$ for all n .

The derived category $D(A)$ is triangulated, the suspension functor Σ being the left shift of a complex together with a sign change in the differential. Given complexes M and N as above, for each n we may identify the cohomology group $\text{Ext}_A^n(M, N)$ with $\text{Hom}_{D(A)}(M, \Sigma^n N)$. For complexes X and Y in $D(A)$, denote the graded A -module $\bigoplus_{n=0}^{\infty} \text{Ext}_A^n(X, Y)$ by $\text{Ext}_A^*(X, Y)$. Using composition of maps in $D(A)$, the graded A -module $\text{Ext}_A^*(M, M)$ becomes a ring, and $\text{Ext}_A^*(M, N)$ becomes a graded right $\text{Ext}_A^*(M, M)$ -module.

We end this section by recalling the notion of thickenings in an arbitrary triangulated category \mathcal{T} with a suspension functor Σ (cf. [11]). A subcategory of \mathcal{T} is *thick* if it is a full triangulated subcategory closed under direct summands. Now let \mathcal{C} and \mathcal{D} be subcategories of \mathcal{T} . We denote by $\text{thick}_{\mathcal{T}}^1(\mathcal{C})$ the full subcategory of \mathcal{T} consisting of all the direct summands of finite direct sums of shifts of objects in \mathcal{C} . Furthermore, we denote by $\mathcal{C} * \mathcal{D}$ the full subcategory of \mathcal{T} consisting of objects M such that there exists a distinguished triangle

$$C \rightarrow M \rightarrow D \rightarrow \Sigma C$$

in \mathcal{T} , with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Now for each $n \geq 2$, define inductively $\text{thick}_{\mathcal{T}}^n(\mathcal{C})$ to be $\text{thick}_{\mathcal{T}}^1(\text{thick}_{\mathcal{T}}^{n-1}(\mathcal{C}) * \text{thick}_{\mathcal{T}}^1(\mathcal{C}))$, and denote $\bigcup_{n=1}^{\infty} \text{thick}_{\mathcal{T}}^n(\mathcal{C})$ by $\text{thick}_{\mathcal{T}}(\mathcal{C})$. This is the smallest thick subcategory of \mathcal{T} containing \mathcal{C} . For example, if A is any ring, then $\text{thick}_{D(A)}(A)$ is the thick subcategory of $D(A)$ formed by the perfect complexes.

3. COMPLEXES OF FINITE CI-DIMENSION

Let A be a local ring. Recall that a *quasi deformation* of A is a diagram $A \rightarrow R \leftarrow Q$ of local homomorphisms, in which $A \rightarrow R$ is faithfully flat, and $R \leftarrow Q$ is surjective with kernel generated by a regular sequence. A homologically finite complex $M \in D(A)$ has finite *complete intersection dimension* if there exists such

a quasi deformation for which the complex $R \otimes_A M$ is perfect over Q . From now on, we write “CI-dimension” instead of “complete intersection dimension”.

The notion of CI-dimension was first introduced for modules in [3], and adopted to complexes in [12]. The terminology reflects the fact that a local ring is a complete intersection precisely when all its finitely generated modules have finite CI-dimension (cf. [3, Theorem 1.3]). The same holds if we replace “modules” with “homologically finite complexes”.

In order to prove the main result, we need the following lemma. It shows that finite CI-dimension is preserved in thick subcategories.

Lemma 3.1. *Let A be a local ring, and let $M \in D(A)$ be a homologically finite complex of finite CI-dimension. Then every complex in $\text{thick}_{D(A)}(M)$ has finite CI-dimension.*

Proof. Let $A \rightarrow R \leftarrow Q$ be a quasi-deformation of A such that the complex $R \otimes_A M$ is perfect over Q . We show by induction that for all $n \geq 1$, every complex $X \in \text{thick}_{D(A)}^n(M)$ satisfies $\text{pd}_Q(R \otimes_A X) < \infty$. If X is a direct summand of finite direct sums of shifts of M , then this clearly holds. Therefore $\text{pd}_Q(R \otimes_A X)$ is finite for all complexes $X \in \text{thick}_{D(A)}^1(M)$. Next, suppose the claim holds for all $1, \dots, n$, and let X be a complex in $\text{thick}_{D(A)}^n(M) * \text{thick}_{D(A)}^1(M)$. Then there exists a triangle

$$Y \rightarrow X \rightarrow Z \rightarrow \Sigma Y$$

in $D(A)$, where Y and Z are complexes in $\text{thick}_{D(A)}^n(M)$ and $\text{thick}_{D(A)}^1(M)$, respectively. By induction, $\text{pd}_Q(R \otimes_A Y)$ and $\text{pd}_Q(R \otimes_A Z)$ are both finite, hence so is $\text{pd}_Q(R \otimes_A X)$. Since

$$\text{thick}_{D(A)}^{n+1}(M) = \text{thick}_{D(A)}^1 \left(\text{thick}_{D(A)}^n(M) * \text{thick}_{D(A)}^1(M) \right),$$

the proof is complete. \square

Next, we prove the main result; if M is a complex of finite CI-dimension and complexity c , then every $0 \leq t \leq c$ “occurs” as a complexity in the thick subcategory generated by M .

Theorem 3.2. *Let A be a local ring, and let $M \in D(A)$ be a nonzero homologically finite complex of finite CI-dimension. Then for every $0 \leq t \leq \text{cx } M$, there exists a nonzero complex in $\text{thick}_{D(A)}(M)$ of complexity t .*

Proof. If $c = 0$, then there is nothing to prove, so suppose that c is nonzero. By Lemma 3.1, it suffices to construct a nonzero complex in $\text{thick}_{D(A)}(M)$ of complexity $c - 1$.

Let $Z_A(M)$ be the graded A -subalgebra of $\text{Ext}_A^*(M, M)$ generated by the central elements in degree two. We first show that the $Z_A(M)$ -module $\text{Ext}_A^*(M, k)$ is Noetherian, by adopting the arguments from the proof of [3, Theorem 4.9]. Let $A \rightarrow R \leftarrow Q$ be a quasi-deformation of A such that $\text{pd}_Q(R \otimes_A M)$ is finite, and let $Z_R(R \otimes_A M)$ be the graded R -subalgebra of $\text{Ext}_R^*(R \otimes_A M, R \otimes_A M)$ generated by the central elements in degree two. By [5, Corollary 5.1], the $Z_R(R \otimes_A M)$ -module $\text{Ext}_R^*(R \otimes_A M, R \otimes_A k)$ is Noetherian, where k is the residue field of A . Moreover, by [3, Theorem 4.9], we may identify $Z_R(R \otimes_A M)$ with $R \otimes_A Z_A(M)$, and so by faithfully flat descent the $Z_A(M)$ -module $\text{Ext}_A^*(M, k)$ is Noetherian (see also [4, Section 7]).

By [7, Lemma 2.5], there exists a positive degree element $\eta \in Z_A(M)$ with the property that scalar multiplication

$$\text{Ext}_A^n(M, k) \xrightarrow{\cdot \eta} \text{Ext}_A^{n+|\eta|}(M, k)$$

is injective for $n \gg 0$. This element corresponds to a map $M \xrightarrow{f_\eta} \Sigma^{|\eta|} M$ in $D(A)$, and completing this map we obtain a triangle

$$M \xrightarrow{f_\eta} \Sigma^{|\eta|} M \rightarrow K \rightarrow \Sigma M$$

in $\text{thick}_{D(A)}(M)$. Note that the object K is nonzero; if not, then M would be isomorphic to $\Sigma^{|\eta|} M$, and this is impossible since $|\eta| \geq 1$ and the nonzero complex M is homologically finite. The triangle induces a long exact sequence

$$\cdots \rightarrow \text{Ext}_A^n(K, k) \rightarrow \text{Ext}_A^{n-|\eta|}(M, k) \xrightarrow{\cdot(-1)^{n\eta}} \text{Ext}_A^n(M, k) \rightarrow \text{Ext}_A^{n+1}(K, k) \rightarrow \cdots$$

in cohomology, hence for some integer n_0 the equality $\beta_{n+1}(K) = \beta_n(M) - \beta_{n-|\eta|}(M)$ holds for $n \geq n_0$. The Poincaré series of K is then given by

$$\begin{aligned} P(K, t) &= \sum_{n=0}^{\infty} \beta_n(K) t^n \\ &= \sum_{n=0}^{n_0} \beta_n(K) t^n + \sum_{n=n_0+1}^{\infty} \beta_n(K) t^n \\ &= \sum_{n=0}^{n_0} \beta_n(K) t^n + \sum_{n=n_0+1}^{\infty} (\beta_{n-1}(M) - \beta_{n-|\eta|-1}(M)) t^n \\ &= g(t) + (t - t^{|\eta|+1}) P(M, t) \\ &= g(t) + t(1 + t + \cdots + t^{|\eta|-1})(1 - t) P(M, t), \end{aligned}$$

where $g(t)$ is a polynomial in $\mathbb{Z}[t]$.

By [12, Corollary 3.10(iv)], the Poincaré series $P(M, t)$ of M is rational, hence so is $P(K, t)$. Therefore, the complexities of M and K equal the orders of the poles at $t = 1$ of $P(M, t)$ and $P(K, t)$, respectively. Consequently, the complexity of K is $\text{cx } M - 1$. Moreover, we see from the construction of K that this complex belongs to $\text{thick}_{D(A)}(M)$. This completes the proof. \square

The main idea in the previous proof is also present in the proof of [6, Proposition 6.1]. Namely, the use of iterated mapping cones to build complexes of prescribed complexity. In the above proof, the central elements used to form these mapping cones belong to the centers of the Ext-algebras of the complexes involved. However, when the ring is a complete intersection, these central elements come from the center of the derived category, as in [6].

Using [8, 3.8], the following corollary follows immediately from Theorem 3.2. It settles the question raised in [8, Remarks 9.6]; every nonzero homologically finite complex of finite CI-dimension over a local ring is virtually small. In other words, the thick subcategory it generates contains a nonzero perfect complex.

Corollary 3.3. *Let A be a local ring, and let $M \in D(A)$ be a nonzero homologically finite complex of finite CI-dimension. Then*

$$\text{thick}_{D(A)}(M) \cap \text{thick}_{D(A)}(A)$$

is nonzero.

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