ON THE EXISTENCE OF CLUSTER TILTING OBJECTS IN TRIANGULATED CATEGORIES

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Abstract. We show that in a triangulated category, the existence of a cluster tilting object often implies that the homomorphism groups are bounded in size. This holds for the stable module category of a selfinjective algebra, and as a corollary we recover a theorem of Erdmann and Holm. We then apply our result to Calabi-Yau triangulated categories, in particular stable categories of maximal Cohen-Macaulay modules over commutative local complete Gorenstein algebras with isolated singularities. We show that the existence of almost all kinds of cluster tilting objects can only occur if the algebra is a hypersurface.

1. Introduction

Cluster categories of finite dimensional hereditary algebras were introduced in [BMRRT] and [CCS], the latter treating the $A_n$ case. These categories are defined as certain orbit categories of the derived categories of modules. A result of Keller (cf. [Kel]) shows that they are triangulated, and moreover they are $2$-Calabi Yau. They were introduced as a representation theoretic categorification of the combinatorics of the cluster algebras introduced by Fomin and Zelevinsky in [FoZ].

Cluster tilting objects were introduced in [BMRRT, Iya, KeR] in order to generalize the classical tilting theory for hereditary algebras. In a cluster category, such objects always exist, for example the stalk complex formed by the underlying hereditary algebra. However, the notion of a cluster tilting object makes sense for any triangulated category, and it is therefore natural to ask the following:

Question. Which triangulated categories contain cluster tilting objects?

We show in this paper that the existence of a cluster tilting object often implies that there is a bound on the size of the homomorphism groups between objects in the category. In Section 2, we prove that this applies to stable module categories of selfinjective algebras. As a corollary, we recover Erdmann and Holm’s result [ErH, Theorem 1.1]: if a cluster tilting object exists for such an algebra, then the minimal projective resolution of every module is bounded. In Section 3, we focus on Calabi-Yau categories, in particular stable categories of maximal Cohen-Macaulay modules over commutative local complete Gorenstein algebras with isolated singularities. We show that the existence of almost all kinds of cluster tilting objects can only occur if the algebra is a hypersurface.

2. Cluster Tilting and Serre Duality

We start by defining cluster tilting objects, or, more precisely, $n$-cluster tilting objects.
Definition. Let \((T, \Sigma)\) be a triangulated category and \(n\) a positive integer. An object \(M\) in \(T\) is \(n\)-cluster tilting if

\[
\text{add}(M) = \{N \in T \mid \text{Hom}_T(M, \Sigma^i N) = 0 \text{ for } 1 \leq i \leq n - 1\} = \{N \in T \mid \text{Hom}_T(N, \Sigma^i M) = 0 \text{ for } 1 \leq i \leq n - 1\}.
\]

When \(n = 1\), the vanishing conditions in the definition are empty, and they are trivially satisfied by all the objects in the category. Therefore, a 1-cluster tilting object is an object \(M\) with \(\text{add}(M) = T\). If \(T\) is a Krull-Schmidt category, which will be the case in our applications, then such a 1-cluster tilting object exists if and only if the category contains only finitely many isomorphism classes of indecomposable objects.

Suppose now that \(k\) is a field and \((T, \Sigma)\) a triangulated Hom-finite \(k\)-category. Thus \(\text{Hom}_T(M, N)\) is a finite dimensional \(k\)-vector space for all objects \(M, N\), and composition of morphisms in \(T\) is \(k\)-bilinear. A Serre functor on \(T\) is an equivalence \(S : T \to T\) of triangulated \(k\)-categories, together with functorial isomorphisms

\[
\text{Hom}_T(M, N) \cong D \text{Hom}_T(N, SM)
\]

of \(k\)-vector spaces for all objects \(M, N \in T\) (where \(D\) denotes the \(k\)-vector space dual). If a triangulated category admits a Serre functor, then by [BoK] it is unique up to canonical isomorphism of triangle functors, and the category is said to have Serre duality.

The following result shows that if a Krull-Schmidt triangulated category \((T, \Sigma)\) with a Serre functor \(S\) contains an \(n\)-cluster tilting object, then every indecomposable summand of that object is \(\Sigma^{-n}\)-periodic.

**Proposition 2.1.** Let \(k\) be a field and \((T, \Sigma)\) a Hom-finite triangulated Krull-Schmidt \(k\)-category with a Serre functor \(S : T \to T\). Furthermore, let \(M\) be an \(n\)-cluster tilting object in \(T\) for some \(n \geq 1\), and \(N\) an indecomposable summand of \(M\). Then \(N \cong \Sigma^{-n} N\) for some \(t \geq 1\).

**Proof.** Suppose first that \(n \geq 2\). By definition \(\text{Hom}_T(N, \Sigma^{-i} M) = 0\) for \(1 \leq i \leq n - 1\), and so the Serre duality gives

\[
0 = \text{Hom}_T(N, \Sigma^{-i} M) \cong D \text{Hom}_T(\Sigma^{-i} M, SN) = D \text{Hom}_T(M, \Sigma^{i}(\Sigma^{-n} N))
\]

for \(1 \leq i \leq n - 1\). The object \(\Sigma^{-n} N\) therefore belongs to \(\text{add}(M)\), and similarly the object \(\Sigma^{-1} \Sigma^{-n} N\) belongs to \(\text{add}(M)\). This implies that all the indecomposable objects

\[
\{(\Sigma^{-n})^i N \mid i \in \mathbb{Z}\}
\]

belong to \(\text{add}(M)\). This is also the case when \(n = 1\), since in this case \(\text{add}(M) = T\).

Since \(T\) is a Krull-Schmidt category, there are only finitely many indecomposable objects in \(\text{add}(M)\), namely the indecomposable summands of \(M\). Hence there exists an integer \(t \geq 1\) such that \((\Sigma^{-n})^i N\) is isomorphic to \(N\).

We shall show that in many cases, cluster tilting objects can only exist if the rate of growth of the cohomology of the category is at most one. To be precise, let \(b_1, b_2, \ldots\) be a sequence of non-negative integers. The rate of growth of the sequence, denoted \(\gamma((b_i)_{i=1}^\infty)\), is defined as

\[
\gamma((b_i)_{i=1}^\infty) = \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ with } b_n \leq an^{t-1} \text{ for } n \gg 0\}.
\]

It measures the polynomial rate of growth (if any) of the sequence. Now let \(k\) be a field and \((T, \Sigma)\) a Hom-finite triangulated \(k\)-category. Given two objects \(M, N \in T\), we define the positive complexity of the ordered pair \((M, N)\) as

\[
\text{cx}_T^+(M, N) = \gamma((\text{dim}_k \text{Hom}_T(M, \Sigma^n N))_{n=1}^\infty),
\]
and the negative complexity as
\[ \text{cx}_M^-(M, N) = \gamma \left( \left( \dim_k \text{Hom}_T(M, \Sigma^{-n} N) \right)_{n=1}^\infty \right). \]

These need not be finite. By definition, \( \text{cx}_M^+(M, N) = 0 \) if and only if \( \text{Hom}_T(M, \Sigma^n N) = 0 \) for \( n > 0 \), whereas \( \text{cx}_M^+(M, N) = 1 \) if and only if the sequence
\[ \dim_k \text{Hom}_T(M, \Sigma N), \dim_k \text{Hom}_T(M, \Sigma^2 N), \dim_k \text{Hom}_T(M, \Sigma^3 N), \ldots \]
is bounded and non-zero infinitely often. Note that the order of the objects \( M \) and \( N \) matters: there is in general no reason to expect that \( \text{cx}_M^+(M, N) = \text{cx}_N^+(N, M) \).

Note also that the positive complexity of a pair is invariant under arbitrary suspension: if \( a \) and \( b \) are any integers, then \( \text{cx}_M^+(M, N) = \text{cx}_M^+(\Sigma^a M, \Sigma^b N) \). Of course, all of this also applies to the negative complexity.

What we shall show is that for certain classes of triangulated categories, for cluster tilting objects to exist it is necessary that the positive and negative complexity of any pair of objects is at most one. The proof relies on the fact that in any triangulated category \( (T, \Sigma) \), a cluster tilting object \( M \) generates the category in the sense that thick\(_T(M) = T \) (cf. [BeO, Lemma 2.2]), where thick\(_T(M) \) denotes the smallest thick subcategory of \( T \) containing \( M \).

The first class of categories we deal with are stable module categories over selfinjective algebras. Let \( k \) be a field and \( \Lambda \) a finite dimensional selfinjective \( k \)-algebra. We denote the stable module category of finitely generated left \( \Lambda \)-modules by \( \text{mod}_\Lambda \).

Every finite dimensional algebra is Morita equivalent to a basic algebra, hence we may assume that \( \Lambda \) is basic. It then follows from [SkY, Chapter IV, Proposition 3.9] that \( \Lambda \) is a Frobenius algebra, i.e. isomorphic as a left module (or, equivalently, as a right module) to \( D(\Lambda) \). Let \( \nu: \Lambda \to \Lambda \) be a Nakayama automorphism of \( \Lambda \), and for a module \( M \), denote by \( \nu M \) the corresponding twisted module. As a vector space, this module coincides with \( M \), but scalar multiplication is given by \( \lambda \cdot m = \nu(\lambda)m \) for \( \lambda \in \Lambda \) and \( m \in M \). By [SkY, Chapter IV, Proposition 3.13], the equivalences \( D \text{Hom}_\Lambda(-, \Lambda) \) and \( \nu(-) \) on \( \text{mod}_\Lambda \) are naturally isomorphic, hence the Auslander-Reiten translate \( \tau = D\text{Tr} \) is naturally isomorphic to \( \Omega_\Lambda^2 \circ \nu(-) \).

By the above, the Auslander-Reiten formula (cf. [AuR]) takes the form
\[ \text{Hom}_\Lambda(M, N) \simeq D \text{Hom}_\Lambda(N, \Omega_\Lambda(\nu M)) \]
for modules \( M \) and \( N \). Consequently, the equivalence \( \Omega_\Lambda \circ \nu(-) \) is a Serre functor on \( \text{mod}_\Lambda \). Now let \( M \) be an \( n \)-cluster tilting object, and \( N \) an indecomposable summand. It follows from Proposition 2.1 that there exists a positive integer \( t \geq 1 \) with \( N \simeq \Omega_\Lambda^{n+1}(\nu t N) \), and since this holds for all the summands of \( M \), there exist
positive integers \( r, s \geq 1 \) such that \( M \simeq \Omega_A^n(M) \). This gives \( \Omega_A^i(M) \simeq \Omega_A^{i+1}(\nu^s M) \) for all integers \( i \in \mathbb{Z} \), and so
\[
\dim_k \Omega_A^i(M) = \dim_k \Omega_A^{i+1}(M)
\]
for all \( i \in \mathbb{Z} \). For all modules \( X, Y \) there is an inequality \( \dim_k \text{Hom}_A(X, Y) \leq (\dim_k X)(\dim_k Y) \), hence both the sequences
\[
(\dim_k \text{Hom}_A(M, \Omega_A^{-i}(M)))_{i=1}^\infty, \quad (\dim_k \text{Hom}_A(M, \Omega_A^i(M)))_{i=1}^\infty
\]
are bounded. By definition, this implies that both \( \text{cx}^+_{\text{mod} A}(M, M) \) and \( \text{cx}^-_{\text{mod} A}(M, M) \) are at most one.

Finally, by [BeO, Lemma 2.2], the thick subcategory of \( \text{mod} A \) generated by \( M \) is \( \text{mod} \Lambda \) itself. The result therefore follows from [BeO, Lemma 2.1].

As a consequence of the theorem, we recover a result of Erdmann and Holm on complexity of modules over a selfinjective algebra. Recall that for a finite dimensional algebra, the complexity of a finitely generated left module measures the polynomial rate of growth (if any) of the dimensions of the modules in its minimal projective resolution. More precisely, let \( \Lambda \) be a finite dimensional algebra over a field \( k \), and \( M \) a finitely generated left module with a minimal free resolution
\[
\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.
\]
Then the complexity of \( M \), denoted \( \text{cx}_A M \), is defined as the rate of growth of the sequence \((\dim_k P_n)_{n=1}^\infty \); that is
\[
\text{cx}_A M \overset{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ with } \dim_k P_n \leq an^{t-1} \text{ for } n \geq 0\}.
\]
As with the category version of complexity, the complexity of a module need not be finite. Also, note that the complexity is zero if and only if \( M \) has finite projective dimension, and \( \text{cx}_A M = 1 \) if and only if the sequence \((\dim_k P_n)_{n=1}^\infty \) is bounded and non-zero infinitely often. The latter happens for example when \( M \) is periodic, that is, when \( M \simeq \Omega_A^i(M) \) for some \( t \geq 1 \). However, there exist finite dimensional algebras, even selfinjective ones, admitting non-periodic modules of complexity one: the first such example appeared in [Sch].

**Corollary 2.3.** [ErH, Theorem 1.1] Let \( k \) be a field and \( \Lambda \) a finite dimensional selfinjective \( k \)-algebra. If \( \text{mod} \Lambda \) contains an \( n \)-cluster tilting object for some \( n \geq 1 \), then \( \text{cx}_A M \leq 1 \) for every finitely generated left \( \Lambda \)-module \( M \).

**Proof.** Let \( \tau \) be the Jacobson radical of \( \Lambda \). It is well-known (see, for instance, [Be1, Section 3]) that the complexity of \( M \) equals the rate of growth of the sequence \((\text{Ext}^n_A(M, \Lambda/\tau))_{n=1}^\infty \). The latter equals \( \text{cx}^+_{\text{mod} A}(M, \Lambda/\tau) \), hence the result follows from Theorem 2.2. \( \square \)

### 3. Cluster tilting and Calabi-Yau categories

In this section, we turn to triangulated categories with particularly nice Serre functors. Let \( k \) be a field and \( (T, \Sigma) \) a triangulated \( \text{Hom} \)-finite \( k \)-category. Then \( T \) is \( d \)-**Calabi-Yau** if it admits a Serre functor which is isomorphic as a \( k \)-linear triangle functor to \( \Sigma^d \). In this case, for all objects \( M, N \in T \), there is an isomorphism
\[
\text{Hom}_T(M, N) \simeq D \text{Hom}_T(N, \Sigma^d M)
\]
of \( k \)-vector spaces. Note that \( d \) can be any integer: it is called the **Calabi-Yau dimension** of the category.
Remark 3.1. If $(T, \Sigma)$ is a non-zero $d$-Calabi-Yau triangulated category, and $d \geq 1$, then an $n$-cluster tilting object cannot exist for $n \geq d + 1$. For if $M$ was such an object, then in particular $\text{Hom}_T(M, \Sigma^d M)$ would be zero, and then the $d$-Calabi-Yau property would imply that $\text{Hom}_T(M, M) = 0$. But then $M$ itself would be zero, and since $\text{thick}_T(M) = T$ by [BeO, Lemma 2.2], this would mean that $T = 0$, a contradiction. Consequently, for (non-zero) $d$-Calabi-Yau triangulated categories with $d \geq 1$, it only makes sense to consider $n$-cluster tilting objects for $1 \leq n \leq d$.

We now apply Proposition 2.1 to Calabi-Yau categories. As the result shows, in all cases but one the indecomposable summands of a cluster tilting object must be periodic with respect to the suspension.

Proposition 3.2. Let $k$ be a field and $(T, \Sigma)$ a Hom-finite $d$-Calabi-Yau triangulated Krull-Schmidt $k$-category. Furthermore, let $M$ be an $n$-cluster tilting object in $T$ for some $n \geq 1$, and $N$ an indecomposable summand of $M$. Then $N \simeq \Sigma^{r(d-n)} N$ for some $t \geq 1$. If $n = d = 1$, then $N$ is still $\Sigma$-periodic, i.e. $N \simeq \Sigma^r N$ for some $r \geq 1$.

Proof. The first statement follows directly from Proposition 2.1 and the definition of $d$-Calabi-Yau categories. For the second, assume that $N$ is an indecomposable summand of a 1-cluster tilting object $M$. In this case $\text{add}(M) = T$, so the category contains only finitely many indecomposable objects. Since all the objects

$$\{\Sigma^i N \mid i \in \mathbb{Z}\}$$

are indecomposable, the object $N$ must be isomorphic to $\Sigma^r N$ for some $r \geq 1$. □

As a consequence of this result, we show next that in almost all cases, the positive and negative complexities of a pair of objects in a Calabi-Yau category with tilting objects are at most one.

Theorem 3.3. Let $k$ be a field and $(T, \Sigma)$ a Hom-finite $d$-Calabi-Yau triangulated Krull-Schmidt $k$-category. Furthermore, suppose that $T$ admits an $n$-cluster tilting object for some $n \geq 1$. If either $n \neq d$ or $n = d = 1$, then

$$\text{cx}_T^+(M, N) \leq 1$$

$$\text{cx}_T^-(M, N) \leq 1$$

for all objects $M, N \in T$.

Proof. Let $X$ be an $n$-cluster tilting object in $T$. It follows from the assumptions and Proposition 3.2 that every indecomposable summand of $X$ is $\Sigma$-periodic, hence so is $X$ itself. In other words, there exists a positive integer $t \geq 1$ such that $X \simeq \Sigma^t X$. It follows that

$$\text{cx}_T^+(X, X) = \text{cx}_T^-(X, X) = 1$$

(strictly speaking, these complexities could be zero, but this happens only if $X = 0$, and as in the remark prior to Proposition 3.2 this would mean that $T = 0$). Since $\text{thick}_T(X) = T$ by [BeO, Lemma 2.2], we see from [BeO, Lemma 2.1] that

$$\text{cx}_T^+(M, N) \leq 1$$

$$\text{cx}_T^-(M, N) \leq 1$$

for all objects $M, N \in T$. □

The theorem does not cover the case when $n = d \geq 2$, and the underlying reason for this is apparent from Proposition 2.1 and its Calabi-Yau version, Proposition 3.2. Namely, if $M$ is an $n$-cluster tilting object in a Hom-finite $d$-Calabi-Yau Krull-Schmidt triangulated $k$-category $(T, \Sigma)$, then it follows from those results that $M \simeq \Sigma^{t(d-n)} M$ for some $t \geq 1$. Consequently, when $n \neq d$ then $M$ is $\Sigma$-periodic,
and this is precisely the key to the proof of Theorem 3.3. However, when \( n = d \), then the isomorphism \( M \cong \Sigma^{d(n-d)} M \) boils down to \( M \cong M \). Therefore, in this situation Proposition 3.2 is of no use, except for in the special case \( n = d = 1 \).

In fact, there are numerous examples of \( d \)-Calabi-Yau categories with \( d \geq 2 \), containing \( d \)-cluster tilting objects, but in which the positive and negative complexities of pairs of objects can exceed one. Some of the classical cluster categories provide examples with Calabi-Yau dimension two.

**Example.** Let \( k \) be an algebraically closed field and \( H \) a finite dimensional hereditary \( k \)-algebra. Then the bounded derived category \( D^b(\text{mod } H) \) has Auslander-Reiten triangles (cf. [Hap]), and a corresponding auto equivalence \( \tau : D^b(\text{mod } H) \to D^b(\text{mod } H) \). Denote the suspension in \( D^b(\text{mod } H) \) by \([1]\): this is the left shift of a complex. The cluster category of \( H \), denoted \( C_H \), is defined as the orbit category \( D^b(\text{mod } H)/\tau^{-1} \circ [1] \). The objects are the same as in \( D^b(\text{mod } H) \), and the suspension \( \Sigma \) is induced by the shift \([1]\) and is equal to \( \tau \). For objects \( M, N \in C_H \), one defines

\[
\text{Hom}_{C_H}(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D^b(\text{mod } H)}(M, \tau^{-n} N[n]),
\]

and this is a finite dimensional \( k \)-vector space since \( H \) is hereditary. By [Kel], the cluster category \( (C_H, \Sigma) \) is triangulated. Moreover, it follows from [BMRRT, Proposition 1.2 and Proposition 1.7] that this category is Krull-Schmidt, and that there is an isomorphism

\[
\text{Hom}_{C_H}(M, N) = \text{DHom}_{C_H}(N, \Sigma^2 M)
\]

for all objects \( M, N \in C_H \). In short: the category \( (C_H, \Sigma) \) is a Hom-finite \( 2 \)-Calabi-Yau triangulated Krull-Schmidt \( k \)-category.

By [BMRRT], the cluster category \( C_H \) always contains 2-cluster tilting objects. In fact, the stalk complex with \( H \) in degree zero is one example. If \( M \) is any 2-cluster tilting object in \( C_H \), then it follows from [BeO, Lemma 2.1 and Lemma 2.2] that \( \text{cx}^+_{C_H}(M, M) \) and \( \text{cx}^-_{C_H}(M, M) \) are the maximal positive and negative complexities obtained in \( C_H \). Consequently, from [BeO, Theorem 3.2] we see that

\[
\text{cx}^+_{C_H}(M, M) = \text{cx}^-_{C_H}(M, M) = \begin{cases} 1 & \text{if } H \text{ has finite representation type,} \\ 2 & \text{if } H \text{ has tame representation type,} \\ \infty & \text{if } H \text{ has wild representation type.} \end{cases}
\]

Returning to the general theory, we shall now apply Theorem 3.3 to some special Calabi-Yau categories arising from local (meaning commutative Noetherian local) rings. Let \( k \) be an algebraically closed field and \((R, \mathfrak{m}, k)\) a complete \( d \)-dimensional local Gorenstein \( k \)-algebra (a local ring is Gorenstein if it has finite injective dimension as a module over itself). Denote by \( \text{MCM}(R) \) the category of finitely generated maximal Cohen-Macaulay \( R \)-modules, i.e.

\[
\text{MCM}(R) = \{ M \in \text{mod } R \mid \text{Ext}^i_R(M, R) = 0 \text{ for } i \geq 1 \}.
\]

This is a Frobenius category, that is, a Quillen-exact category with enough projective and injective objects that coincide. The stable category \( \text{MCM}(R) \) is therefore a triangulated \( k \)-category, with the cosyzygy functor \( \Omega^1_R : \text{MCM}(R) \to \text{MCM}(R) \) as suspension (cf. [Hap]). Suppose now that \( R \) in addition is an isolated singularity, meaning that \( \mathfrak{p} \) is a regular local ring for every non-maximal prime ideal \( \mathfrak{p} \) of \( R \). Then it is not hard to see that the stable category \( \text{MCM}(R) \) is Hom-finite, and it follows from [Aus, Proposition I.8.8 and Proposition III.1.8] (see also [IyY, Theorem 8.3]) that this triangulated category is in fact \((d-1)\)-Calabi-Yau. The completeness of the ring \( R \) ensures that \( \text{mod } R \) is a Krull-Schmidt category, hence so is \( \text{MCM}(R) \).
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**Question.** For which complete local Gorenstein algebras do the stable categories of maximal Cohen-Macaulay modules admit cluster tilting objects?

This question has been answered for only a few classes of Gorenstein algebras, in particular the following two.

**Examples.** (1) Let $k$ be an algebraically closed field of characteristic zero, and denote by $S$ the power series ring $k[x_1, \ldots, x_d]$ in $d$-variables, with $d \geq 2$. Furthermore, let $G$ be a subgroup of $\text{SL}(d,k)$, and denote the corresponding invariant subring $S^G$ of $S$ by $R$. By a classical result of Watanabe (cf. [Wat]), this ring is a $d$-dimensional (and complete) Gorenstein $k$-algebra. Suppose now in addition that all the elements of $G$, except the identity, do not have 1 as an eigenvalue. Then $R$ is an isolated singularity by [IY, Corollary 8.2]. By [Iya, Theorem 2.5] (see also [IY, Theorem 8.4]), the $R$-module $S$ is a $(d-1)$-cluster tilting object in $\text{MCM}(R)$.

(2) Let $k$ be an algebraically closed field and $(R, \mathfrak{m}, k)$ a complete $d$-dimensional local hypersurface $k$-algebra, i.e. $R \cong S/(f)$ for some local ring $S$ and $f \in S$. Then all the maximal Cohen-Macaulay $R$-modules are two-periodic (cf. [Eis]), hence $\Omega^2_R \cong \Omega_R^{-2} \cong 1$ on $\text{MCM}(R)$. If $R$ is also an isolated singularity, then we already know that the stable category is $(d-1)$-Calabi-Yau. Thus the two-periodicity implies that $\text{MCM}(R)$ is actually 1-Calabi-Yau when $d$ is even, and 2-Calabi-Yau when $d$ is odd. In view of Remark 3.1, when $d$ is even then $\text{MCM}(R)$ cannot contain $n$-cluster tilting objects for $n \geq 2$. Also, by definition, a 1-cluster tilting object exists if and only if $R$ has finite Cohen-Macaulay representation type.

Suppose that the characteristic of $k$ is zero and that $R$ is a simple hypersurface singularity. Then the following hold:

(i) $R \cong k[[x, y, z_1, \ldots, z_{d-1}]]/(f)$ and $f$ is one of the following polynomials:

- $A_n : x^2 + y^{n+1} + z_1^2 + \cdots + z_{d-1}^2$ for $n \geq 1$,
- $D_n : x^2y + y^{n-1} + z_1^2 + \cdots + z_{d-1}^2$ for $n \geq 4$,
- $E_6 : x^3 + y^3 + z_1^2 + \cdots + z_{d-1}^2$,
- $E_7 : x^3 + xy^3 + z_1^2 + \cdots + z_{d-1}^2$,
- $E_8 : x^3 + y^3 + z_1^2 + \cdots + z_{d-1}^2$.

(ii) The Cohen-Macaulay representation type of $R$ is finite.

By (ii), the category $\text{MCM}(R)$ contains a 1-cluster tilting object, and from the above we know that when $d$ is even then no $n$-cluster tilting object exists for $n \geq 2$. When $d$ is odd, then by [BIKR, Theorem 1.3] a 2-cluster tilting object exists in precisely two cases: the $A_n$ case with $n$ odd, and the $D_n$ case with $n$ even.

The second example suggests that the class of complete odd-dimensional hypersurface singularities might provide a source of algebras whose stable categories admit cluster tilting objects. The following result shows that if the stable category of a complete Gorenstein singularity does contain such an object, then in “most cases” it is actually necessary that the the ring be a hypersurface. Recall first that if $M$ is a finitely generated module over (a general) local ring $(R, \mathfrak{m}, k)$, then the complexity of $M$, denoted $\text{cx}_R M$, is the rate of growth of the ranks of the free modules in the minimal free resolution. That is, if

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

is the minimal free resolution of $M$, then

$$\text{cx}_R M \overset{\text{def}}{=} \inf \{ t \in \mathbb{N} \cup \{ 0 \} \mid \exists a \in \mathbb{R} \text{ with } \text{rank } F_n \leq an^{t-1} \text{ for } n \gg 0 \}. $$

Since the rank of the free module $F_n$ equals the dimension of the $k$-vector space $\text{Ext}_R^n(M, k)$, the complexity of $M$ equals the rate of growth of the sequence

$$(\dim_k \text{Ext}_R^n(M, k))_{n=1}^{\infty}.$$
In the local case, complexity characterizes the ring $R$ itself. Namely, in [Gul] Gulliksen proved that the following are equivalent:

1. $R$ is a complete intersection,
2. all finitely generated $R$-modules have finite complexity,
3. $cx_R k$ is finite and equal to the codimension of $R$.

Recall that $R$ being a complete intersection means that the completion of $R$ has a presentation $S/(x_1, \ldots, x_n)$, with $S$ regular local and $x_1, \ldots, x_n$ a regular sequence. In this case, there exists such a presentation with $c$ equal to the codimension of $R$, and by (3) above this is the complexity of the residue field $k$. Also, this is the maximal complexity obtained by the $R$-modules. In particular, if $cx_R k \leq 1$, then $R$ is a hypersurface.

**Theorem 3.4.** Let $k$ be an algebraically closed field and $(R, m, k)$ a complete $d$-dimensional local Gorenstein $k$-algebra with an isolated singularity. Furthermore, suppose that $\text{MCM}(R)$ contains an $n$-cluster tilting object for some $n \geq 1$. Then $R$ is a hypersurface if $n \neq d - 1$ or $n = d - 1 = 1$. If in addition $n \geq 3$, or $d$ is even an $n \geq 2$, then $R$ is a regular local ring.

**Proof.** Our stable category $\text{MCM}(R)$ is a Hom-finite $(d - 1)$-Calabi-Yau triangulated Krull-Schmidt $k$-category. Therefore, if either $n \neq d - 1$ or $n = d - 1 = 1$, then it follows from Theorem 3.3 that

$$cx_{\text{MCM}(R)}^{\ast}(M, N) \leq 1$$

for all maximal Cohen-Macaulay $R$-modules $M$ and $N$.

Denote the bounded derived category of finitely generated $R$-modules by $D^b(\text{mod} R)$, and its thick subcategory of perfect complexes by $D^b_{\text{perf}}(\text{mod} R)$. By [Buc, Theorem 4.4.1], the stable derived category $D^b_{\text{st}}(\text{mod} R)$, which is the Verdier quotient $D^b(\text{mod} R)/D^b_{\text{perf}}(\text{mod} R)$, is equivalent, as a triangulated $k$-category, to $\text{MCM}(R)$. Hence the positive complexity of any pair of objects in $D^b_{\text{st}}(\text{mod} R)$ is at most one. Now consider the $R$-module $k$, viewed as a stalk complex in $D^b(\text{mod} R)$ (with $k$ concentrated in degree zero). By [Buc, Corollary 6.3.4], the natural map

$$\text{Hom}^{D_{\text{st}}(\text{mod} R)}(k, k[t]) \to \text{Hom}^{D_{\text{st}}(\text{mod} R)}(k, k[t])$$

induced by the quotient functor $D^b(\text{mod} R) \to D^b_{\text{st}}(\text{mod} R)$ is an isomorphism of $k$-vector spaces for $t \geq 0$, where $k[t]$ is the $t$’th suspension of the stalk complex $k$. But for $t \geq 1$, the vector spaces $\text{Ext}_{D_{\text{st}}(\text{mod} R)}^r(k, k)$ and $\text{Hom}_{D_{\text{st}}(\text{mod} R)}^r(k, k[t])$ are isomorphic.

Since the complexity of the pair $(k, k)$ in $D^b_{\text{st}}(\text{mod} R)$ is at most one, we therefore see that the rate of growth of the sequence

$$(\dim_k \text{Ext}_{D_{\text{st}}(\text{mod} R)}^n(k, k))^{\frac{1}{n}}$$

is also at most one. Thus $cx_R k \leq 1$, hence $R$ is a hypersurface.

To prove the second statement, suppose that $R$ is a hypersurface and that $M$ is an $n$-cluster tilting object in $\text{MCM}(R)$. Furthermore, suppose that either $n \geq 3$ or that $d$ is even and $n \geq 2$. We saw in the above example that $\text{MCM}(R)$ is always $2$-Calabi-Yau, so if $n \geq 3$ then

$$0 = \text{Hom}_{\text{MCM}(R)}(M, \Omega_R^{-2}(M)) \cong D(\text{Hom}_{\text{MCM}(R)}(M, M)).$$

If $d$ is even then $\text{MCM}(R)$ is $1$-Calabi-Yau, so in this case we obtain

$$0 = \text{Hom}_{\text{MCM}(R)}(M, \Omega_R^{-1}(M)) \cong D(\text{Hom}_{\text{MCM}(R)}(M, M))$$

whenever $n \geq 2$. In both cases $M = 0$, and since by [BeO, Lemma 2.2] the thick subcategory in $\text{MCM}(R)$ generated by $M$ is $\text{MCM}(R)$ itself, we conclude that the category must be trivial. But then all maximal Cohen-Macaulay $R$-modules are free, hence $R$ is a regular local ring. □
We end this paper with a result concerning more general complete intersections. Let \((R, m, k)\) be a local complete intersection of codimension \(c\), say. Then the stable category \(\text{MCM}(R)\) of maximal Cohen-Macaulay modules is not necessarily Hom-finite over a field, Krull-Schmidt nor Calabi-Yau. But it is of course still triangulated. Now for any integer \(1 \leq t \leq c\), consider the following thick subcategory:

\[
\text{MCM}_{\leq t}(R) \overset{\text{def}}{=} \{ M \in \text{MCM}(R) \mid \text{cx}_R M \leq t \}.
\]

To see why it is thick, consider a short exact sequence

\[
0 \to M_1 \to M_2 \to M_3 \to 0
\]

of \(R\)-modules. Then it is easy to show that \(\text{cx}_R M_u \leq \max\{\text{cx}_R M_v, \text{cx}_R M_w\}\) for \(\{u, v, w\} = \{1, 2, 3\}\). Since triangles in \(\text{MCM}(R)\) correspond to short exact sequences, we see that if two objects in a triangle belong to \(\text{MCM}_{\leq t}(R)\), then so does the third. Hence these subcategories are thick. In particular, they are triangulated categories themselves.

We shall show that if \(c \geq 2\), then none of the triangulated subcategories \(\text{MCM}_{\leq 1}(R), \ldots, \text{MCM}_{\leq c-1}(R)\) can contain a cluster tilting object. The proof uses the theory of support varieties developed by Avramov and Buchweitz in [AvB], we recall the basic definition. Let \(\hat{R}\) be the completion of \(R\), and \(\hat{R}[\chi] = \hat{R}[\chi_1, \ldots, \chi_c]\) the polynomial ring in the commuting Eisenbud operators. For every \(\hat{R}\)-module \(X\), there is a ring homomorphism

\[
\hat{R}[\chi] \to \text{Ext}^*_{\hat{R}}(X, X),
\]

under which \(\text{Ext}^*_{\hat{R}}(X, X)\) is a finitely generated \(\hat{R}[\chi]\)-module. Then \(k \otimes_R \text{Ext}^*_{\hat{R}}(X, X)\) is a finitely generated module over the tensor product \(k \otimes_R \hat{R}[\chi]\), which is canonically isomorphic to \(k[\chi]\). The support variety of an \(R\)-module \(M\), denoted \(V_R(M)\), is defined as

\[
V_R(M) \overset{\text{def}}{=} \{ \alpha \in \overline{k}^c \mid f(\alpha) = 0 \text{ for all } f \in \text{Ann}_{k[\chi]} k \otimes_R \text{Ext}^*_{\hat{R}}(M, \hat{M}) \},
\]

where \(\overline{k}\) is the algebraic closure of \(k\).

**Theorem 3.5.** Let \((R, m, k)\) be a local complete intersection of codimension \(c\), with \(c \geq 2\). Then for \(1 \leq t \leq c-1\), the triangulated category

\[
\text{MCM}_{\leq t}(R) = \{ M \in \text{MCM}(R) \mid \text{cx}_R M \leq t \}
\]

do not contain a cluster tilting object.

**Proof.** Let \(1 \leq t \leq c-1\), and suppose that \(\text{MCM}_{\leq t}(R)\) does contain a cluster tilting object \(M\). Consider a short exact sequence

\[
0 \to M_1 \to M_2 \to M_3 \to 0
\]

of \(R\)-modules. Then \(V_R(M_u) \subseteq V_R(M_v), V_R(M_w)\) for \(\{u, v, w\} = \{1, 2, 3\}\) by [AvB, Theorem 5.6]. Since the triangles in \(\text{MCM}_{\leq t}(R)\) correspond to short exact sequences, and \(\text{cx}_{\text{MCM}_{\leq t}(R)}(M) = \text{cx}_{\text{MCM}_{\leq t}(R)}(M)\) by [BeO, Lemma 2.2], we see that \(V_R(N) \subseteq V_R(M)\) for all modules \(N \in \text{MCM}_{\leq t}(R)\). Moreover, the complexity of \(M\) must be \(t\).

By [AvB, Theorem 5.6], the complexity of any \(R\)-module equals the dimension of its support variety. Thus \(\text{dim} V_R(M) = t \leq c-1\). Now choose a one-dimensional closed and homogeneous subspace \(V\) of \(\overline{k}\) with \(V \nsubseteq V_R(M)\). By [Be2, Corollary 2.3], there exists a maximal Cohen-Macaulay \(R\)-module \(N\) with \(V_R(N) = V\). But this is a contradiction: the complexity of \(N\) is one, hence \(N \in \text{MCM}_{\leq 1}(R)\), but \(V_R(N) \nsubseteq V_R(M)\). \(\Box\)
A similar result holds over finite dimensional algebras admitting a theory of support varieties defined in terms of the Hochschild cohomology ring (cf. [EHSST] and [SnS]). Such algebras are Gorenstein, and so Theorem 3.5 holds for the stable category of maximal Cohen-Macaulay modules.

REFERENCES


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