Support via central ring actions

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The notions of central ring actions and support in triangulated categories have proved quite fruitful recently, cf. [1], [2], [3], [4]. They unify ideas and techniques from group cohomology, commutative ring theory (complete intersections) and Hochschild cohomology. This brief survey is an introduction to the basic concepts.

Let $T$ be a triangulated category with suspension functor $\Sigma$. A subcategory of $T$ is thick if it is a full triangulated subcategory closed under direct summands. Given an object $X \in T$, we denote by $\text{thick}_T(X)$ the smallest thick subcategory of $T$ containing $X$; this is the intersection of all thick subcategories containing $X$.

The graded center $Z^*(T)$ of $T$ is a graded ring, whose degree $n$ component $Z^n(T)$ (for $n \in \mathbb{Z}$) consists of the natural transformations $\text{Id} \xrightarrow{\sigma_n} \Sigma^n$ satisfying $f_{\Sigma X} = (-1)^n \Sigma f_X$ on the level of objects. For such a central element $f$ and objects $X, Y \in T$, consider the graded group $\text{Hom}_T^f(X, Y) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_T(X, \Sigma^n Y)$. The element $f$ acts from the right on this graded group via the morphism $X \xrightarrow{fx} \Sigma^n X$, and from the left via the morphism $Y \xrightarrow{\Sigma f} \Sigma^n Y$. Namely, given a morphism $g \in \text{Hom}_T(X, \Sigma^m Y)$, the scalar product $gf$ is the composition $X \xrightarrow{fx} \Sigma^n X \xrightarrow{\Sigma^n g} \Sigma^m Y$, whereas $fg$ is the composition $X \xrightarrow{g} \Sigma^m Y \xrightarrow{\Sigma^m f_y} \Sigma^{m+n} Y$. However, since $\text{Id} \xrightarrow{\Sigma} \Sigma$ is a natural transformation, the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & \Sigma^m Y \\
\downarrow{fx} & & \downarrow{\Sigma f_m Y} \\
\Sigma^n X & \xrightarrow{\Sigma^n g} & \Sigma^{m+n} Y
\end{array}
\]

commutes, and so since $f_{\Sigma^m Y}$ equals $(-1)^m \Sigma^m f_Y$ we see that $gf = (-1)^m fg$. Thus $Z^*(T)$ acts graded-commutatively on $\text{Hom}_T^f(X, Y)$ for all objects $X$ and $Y$ in $T$. For further details on the graded center and its action on the cohomology groups, see [5].

Now let $R = \bigoplus_{n=0}^{\infty} R_n$ be a positively graded ring which is graded-commutative, i.e. $rs = (-1)^{|r||s|}sr$ for all homogeneous elements $r, s \in R$. Then $R$ acts centrally on $T$ if there exists a homomorphism $R \to Z^*(T)$ of graded rings. Thus, for every object $X \in T$, there is a homomorphism $R \xrightarrow{\varphi_X} \text{Hom}_T^*(X, X)$ satisfying the following: for every object $Y$ and all homogeneous elements $r \in R$ and $g \in \text{Hom}_T^*(X, Y)$, the equality

\[g \cdot \varphi_X(r) = (-1)^{|r||g|} \varphi_Y(r) \cdot g\]

holds. In other words, the left and right $R$-module structures on $\text{Hom}_T^*(X, Y)$ coincide up to sign.

Example. Let $k$ be a commutative ring, and let $\Lambda, \Gamma, \Delta$ be $k$-algebras which are projective as $k$-modules. Furthermore, let $\Lambda B_{\Delta}, \Lambda B'_{\Delta}, \Delta M_{\Gamma}, \Delta N_{\Gamma}$ be bimodules with $B$ and $B'$ both $\Delta$-projective. Let $\eta \in \text{Ext}_{\Lambda \otimes_k \Delta^{op}}^1(B, B')$ and $\theta \in \text{Ext}_{\Delta \otimes_k \Gamma^{op}}^1(M, N)$.
be homogeneous elements. Then $B \otimes_{\Delta} \theta$ and $B' \otimes_{\Delta} \theta$ are exact since $B$ and $B'$ are $\Delta$-projective, whereas $\eta \otimes_{\Delta} M$ and $\eta \otimes_{\Delta} N$ are exact since the short exact sequences comprising $\eta$ split as sequences of $\Delta$-modules. It was proved in [6] that the equality

$$(\eta \otimes_{\Delta} N) \circ (B \otimes_{\Delta} \theta) = (-1)^{mn}(B' \otimes_{\Delta} \theta) \circ (\eta \otimes_{\Delta} M)$$

holds, where both sides are elements of $\text{Ext}^{n+1}_{\Delta}(B \otimes_{\Delta} M, B' \otimes_{\Delta} N)$.

Specializing to the case $\Lambda = \Gamma = \Delta = B' = B' = M = N$, we see that the Hochschild cohomology ring $\text{HH}^n(\Lambda) = \bigoplus_{n=0}^\infty \text{Ext}^n_{\Lambda}(\Lambda, \Lambda)$ of $\Lambda$ is graded commutative. Moreover, if $\Lambda = \Delta = B = B'$, $\Gamma = k$ and $M, N$ are left $\Lambda$-modules, then for homogeneous elements $\eta \in \text{HH}^n(\Lambda)$ and $\theta \in \text{Ext}^n_\Lambda(M, N)$ we see that the equality

$$(\eta \otimes_{\Delta} N) \circ \theta = (-1)^{|\theta||\eta|} \cdot (\eta \otimes_{\Lambda} M)$$

holds. Consequently, given a $k$-algebra $\Lambda$ which is $k$-projective, for every left $\Lambda$-module $M$ there is a graded ring homomorphism

$$\text{HH}^*(\Lambda) \xrightarrow{\varphi_M=-\otimes_{\Delta} M} \text{Ext}^*_\Lambda(M, M)$$

satisfying the following: for every left $\Lambda$-module $N$ and all homogeneous elements $\eta \in \text{HH}^*(\Lambda), \theta \in \text{Ext}^*_\Lambda(M, N)$, the equality

$$\varphi_N(\eta) \cdot \theta = (-1)^{|\theta||\eta|} \cdot \varphi_M(\eta)$$

holds. Extending to the derived category $D(\Lambda)$ of $\Lambda$-modules via stalk complexes, we see that $\text{HH}^*(\Lambda)$ acts centrally on $D(\Lambda)$.

Returning to our triangulated category $\mathcal{T}$ and the graded-commutative ring $R$, acting centrally, let $X$ and $Y$ be objects of $\mathcal{T}$. Then the $R$-module $\text{Hom}^*_\mathcal{T}(X, Y)$ is eventually Noetherian, denoted $\text{Hom}^*_\mathcal{T}(X, Y) \in \text{Noeth} R$, if there exists an integer $n_0$ such that the $R$-module $\text{Hom}^{\geq n_0}_\mathcal{T}(X, Y) = \bigoplus_{n=n_0}^\infty \text{Hom}_\mathcal{T}(X, \Sigma^n Y)$ is Noetherian. If, in addition, the $R_0$-module $\text{Hom}_\mathcal{T}(X, \Sigma^n Y)$ has finite length for $n > 0$, then we write $\text{Hom}^*_\mathcal{T}(X, Y) \in \text{Noeth}^R R$ and say that the $R$-module $\text{Hom}^*_\mathcal{T}(X, Y)$ is eventually Noetherian of finite length.

It is not difficult to see that if $\text{Hom}^*_\mathcal{T}(X, Y)$ belongs to Noeth $R$, then it also belongs to Noeth $R^e$, where $R^e$ is the commutative even subring of $\bigoplus_{n=0}^\infty R_{2n}$ of $R$. Similarly, if $\text{Hom}^*_\mathcal{T}(X, Y)$ belongs to Noeth$^R R$, then it also belongs to Noeth$^R R^e$. In the latter case, the rate of growth of the sequence $(\ell_{R_0} \text{Hom}_\mathcal{T}(X, \Sigma^n Y))$ is finite and coincides with the Krull dimension of $\text{Hom}^*_\mathcal{T}(X, Y)$ as an $R^{e}$-module (cf. [4, Proposition 2.6]).

The support of a pair of objects (with respect to $R$) is defined in terms of the homogeneous prime ideals of $R^e$. Denote by $\text{Proj} R^e$ the set of homogeneous prime ideals of $R^e$ not containing $\bigoplus_{n=1}^\infty R_{2n}$. Given two objects $X$ and $Y$ of $\mathcal{T}$, we define the support of the ordered pair $(X, Y)$ as

$$\text{Supp}^+_R(X, Y) \overset{\text{def}}{=} \{ p \in \text{Proj} R^e \mid \text{Hom}^*_\mathcal{T}(X, Y)_p \neq 0 \}.$$  

In the following theorem, we summarize some of the standard elementary properties of support sets (cf. [1]).
Theorem 1 (Properties of support).
(1) \( \text{Supp}_R^+ (X, Y) = \text{Supp}_R^{=n} (X, Y) \) for all \( n \in \mathbb{Z} \).
(2) If \( \text{Hom}^\geq_n (X, Y) \) is a finitely generated \( R \)-module for some \( n \), then
\[
\text{Supp}_R^+ (X, Y) = \{ p \in \text{Proj} R \mid \text{Ann}_R (\text{Hom}^\geq_n (X, Y)) \subseteq p \}.
\]
In particular, if \( \text{Hom}^\geq_n (X, Y) \in \text{Noeth} R \), then \( \text{Supp}_R^+ (X, Y) \) is a closed set in \( \text{Proj} R \).
(3) If \( \text{Hom}^\ast T (X, Y) \in \text{Noeth} R \), then \( \text{Supp}_R^+ (X, Y) \) is empty if and only if \( \text{Hom}^\ast T (X, Y) \) is eventually zero.
(4) Given a triangle \( Z' \rightarrow Z \rightarrow Z'' \rightarrow \Sigma Z' \) in \( T \), there are inclusions
\[
\text{Supp}_R^+ (X, Z) \subseteq \text{Supp}_R^+ (X, Z') \cup \text{Supp}_R^+ (X, Z''),
\]
\[
\text{Supp}_R^+ (Z, Y) \subseteq \text{Supp}_R^+ (Z', Y) \cup \text{Supp}_R^+ (Z'', Y).
\]
(5) If \( G \) is an object in \( T \) with \( \text{thick}^T (G) = T \), then
\[
\text{Supp}_R^+ (X, G) = \text{Supp}_R^+ (X, X) = \text{Supp}_R^+ (G, X).
\]
Properties (3) and (5) provide a criterion for a finite dimensional algebra to be Gorenstein. For such an algebra \( \Lambda \) with radical \( r \), the thick subcategory of \( D^b (\Lambda) \) generated by the stalk complex \( \Lambda/r \) is the whole of \( D^b (\Lambda) \).

Corollary 2. Let \( \Lambda \) be a finite dimensional algebra with radical \( r \). Suppose that \( \text{Ext}^n_\Lambda (\Lambda/r, \Lambda/r) \in \text{Noeth} R \) for some graded-commutative ring \( R \) acting centrally on \( D^b (\Lambda) \) (for example \( R = HH^* (\Lambda) \)). Then for every finitely generated \( \Lambda \)-module \( M \), the implications
\[
pd M < \infty \iff \text{Ext}^n_\Lambda (M, M) = 0 \text{ for } n \gg 0 \iff \text{id} M < \infty
\]
hold. In particular, \( \Lambda \) is Gorenstein.

References