Support via central ring actions

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The notions of central ring actions and support in triangulated categories have proved quite fruitful recently, cf. [1], [2], [3], [4]. They unify ideas and techniques from group cohomology, commutative ring theory (complete intersections) and Hochschild cohomology. This brief survey is an introduction to the basic concepts.

Let $\mathcal{T}$ be a triangulated category with suspension functor Σ. A subcategory of $\mathcal{T}$ is *thick* if it is a full triangulated subcategory closed under direct summands. Given an object $X \in \mathcal{T}$, we denote by $\operatorname{thick}_{\mathcal{T}}(X)$ the smallest thick subcategory of $\mathcal{T}$ containing X; this is the intersection of all thick subcategories containing X.

The *graded center* $Z^*(\mathcal{T})$ of $\mathcal{T}$ is a graded ring, whose degree n component $Z^n(\mathcal{T})$ (for $n \in \mathbb{Z}$) consists of the natural transformations $\operatorname{Id} \xrightarrow{f} \Sigma^n$ satisfying $f_{\Sigma X} = (-1)^n \Sigma f_X$ on the level of objects. For such a central element f and objects $X, Y \in \mathcal{T}$, consider the graded group $\operatorname{Hom}^*_{\mathcal{T}}(X,Y) = \oplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^i Y)$. The element f acts from the right on this graded group via the morphism $X \xrightarrow{f_X} \Sigma^n X$, and from the left via the morphism $Y \xrightarrow{f_Y} \Sigma^n Y$. Namely, given a morphism $g \in \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^m Y)$, the scalar product gf is the composition $X \xrightarrow{f_X} \Sigma^n X \xrightarrow{\Sigma^n g} \Sigma^{m+n} Y$, whereas fg is the composition $X \xrightarrow{g} \Sigma^m Y \xrightarrow{\Sigma^m f_Y} \Sigma^{m+n} Y$. However, since $\operatorname{Id} \xrightarrow{f} \Sigma^n$ is a natural transformation, the diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f_X"] & \Sigma^m Y \arrow[d, "f_{\Sigma^m Y}"] \\
\Sigma^n X \arrow[r, "\Sigma^n g"] & \Sigma^{m+n} Y
\end{tikzcd}$$

commutes, and so since $f_{\Sigma^m Y}$ equals $(-1)^{mn} \Sigma^m f_Y$ we see that $gf = (-1)^{mn} fg$. Thus $Z^*(\mathcal{T})$ acts graded-commutatively on $\operatorname{Hom}^*_{\mathcal{T}}(X,Y)$ for all objects X and Y in $\mathcal{T}$. For further details on the graded center and its action on the cohomology groups, see [5].

Now let $R = \oplus_{n=0}^{\infty} R_n$ be a positively graded ring which is graded-commutative, i.e. $rs = (-1)^{|r||s|} sr$ for all homogeneous elements $r, s \in R$. Then R *acts centrally* on $\mathcal{T}$ if there exists a homomorphism $R \to Z^*(\mathcal{T})$ of graded rings. Thus, for every object $X \in \mathcal{T}$, there is a homomorphism $R \xrightarrow{\varphi_X} \operatorname{Hom}^*_{\mathcal{T}}(X,X)$ satisfying the following: for every object Y and all homogeneous elements $r \in R$ and $g \in \operatorname{Hom}^*_{\mathcal{T}}(X,Y)$, the equality

$$g \cdot \varphi_X(r) = (-1)^{|r||g|} \varphi_Y(r) \cdot g$$

holds. In other words, the left and right R-module structures on $\operatorname{Hom}^*_{\mathcal{T}}(X,Y)$ coincide up to sign.

Example. *Let k be a commutative ring, and let Λ, Γ, Δ be k-algebras which are projective as k-modules. Furthermore, let ${}_{\Lambda}B_{\Delta}, {}_{\Lambda}B'_{\Delta}, {}_{\Delta}M_{\Gamma}, {}_{\Delta}N_{\Gamma}$ be bimodules with B and B' both Δ-projective. Let $\eta \in \operatorname{Ext}^n_{\Lambda \otimes_k \Delta^{\mathrm{op}}}(B, B')$ and $\theta \in \operatorname{Ext}^m_{\Delta \otimes_k \Gamma^{\mathrm{op}}}(M, N)$*

be homogeneous elements. Then $B \otimes_\Delta \theta$ and $B' \otimes_\Delta \theta$ are exact since B and B' are Δ-projective, whereas $\eta \otimes_\Delta M$ and $\eta \otimes_\Delta N$ are exact since the short exact sequences comprising η split as sequences of Δ-modules. It was proved in [6] *that the equality*

$$(\eta \otimes_\Delta N) \circ (B \otimes_\Delta \theta) = (-1)^{mn}(B' \otimes_\Delta \theta) \circ (\eta \otimes_\Delta M)$$

holds, where both sides are elements of $\operatorname{Ext}^{m+n}_{\Lambda \otimes_k \Gamma^{\mathrm{op}}}(B \otimes_\Delta M, B' \otimes_\Delta N)$

Specializing to the case $\Lambda = \Gamma = \Delta = B = B' = M = N$, we see that the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda) = \oplus_{n=0}^{\infty} \operatorname{Ext}^n_{\Lambda \otimes_k \Lambda^{\mathrm{op}}}(\Lambda, \Lambda)$ *of Λ is graded commutative. Moreover, if $\Lambda = \Delta = B = B'$, $\Gamma = k$ and M, N are left Λ-modules, then for homogeneous elements* $\eta \in \mathrm{HH}^*(\Lambda)$ *and* $\theta \in \operatorname{Ext}^*_\Lambda(M, N)$ *we see that the equality*

$$(\eta \otimes_\Lambda N) \circ \theta = (-1)^{|\eta||\theta|}\theta \circ (\eta \otimes_\Lambda M)$$

holds. Consequently, given a k-algebra Λ which is k-projective, for every left Λ-module M there is a graded ring homomorphism

$$\mathrm{HH}^*(\Lambda) \xrightarrow{\varphi_M = - \otimes_\Lambda M} \operatorname{Ext}^*_\Lambda(M, M)$$

satisfying the following: for every left Λ-module N and all homogeneous elements $\eta \in \mathrm{HH}^*(\Lambda), \theta \in \operatorname{Ext}^*_\Lambda(M, N)$*, the equality*

$$\varphi_N(\eta) \cdot \theta = (-1)^{|\eta||\theta|}\theta \cdot \varphi_M(\eta)$$

holds. Extending to the derived category $D(\Lambda)$ of Λ-modules via stalk complexes, we see that $\mathrm{HH}^*(\Lambda)$ *acts centrally on $D(\Lambda)$.*

Returning to our triangulated category $\mathcal{T}$ and the graded-commutative ring R acting centrally, let X and Y be objects of $\mathcal{T}$. Then the R-module $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ is *eventually Noetherian*, denoted $\operatorname{Hom}^*_{\mathcal{T}}(X, Y) \in \operatorname{Noeth} R$, if there exists an integer n_0 such that the R-module $\operatorname{Hom}^{\geq n_0}_{\mathcal{T}}(X, Y) = \oplus_{n=n_0}^{\infty} \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$ is Noetherian. If, in addition, the R_0-module $\operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$ has finite length for $n \gg 0$, then we write $\operatorname{Hom}^*_{\mathcal{T}}(X, Y) \in \operatorname{Noeth}^{\mathrm{fl}} R$ and say that the R-module $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ is *eventually Noetherian of finite length.*

It is not difficult to see that if $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ belongs to $\operatorname{Noeth} R$, then it also belongs to $\operatorname{Noeth} R^{\mathrm{ev}}$, where R^{ev} is the commutative even subring $\oplus_{n=0}^{\infty} R_{2n}$ of R. Similarly, if $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ belongs to $\operatorname{Noeth}^{\mathrm{fl}} R$, then it also belongs to $\operatorname{Noeth}^{\mathrm{fl}} R^{\mathrm{ev}}$. In the latter case, the rate of growth of the sequence $(\ell_{R_0} \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y))$ is finite and coincides with the Krull dimension of $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ as an R^{ev}-module (cf. [4, Proposition 2.6].

The support of a pair of objects (with respect to R) is defined in terms of the homogeneous prime spectrum of R^{ev}. Denote by $\operatorname{Proj} R^{\mathrm{ev}}$ the set of homogeneous prime ideals of R^{ev} not containing $\oplus_{n=1}^{\infty} R_{2n}$. Given two objects X and Y of $\mathcal{T}$, we define the *support* of the ordered pair (X, Y) as

$$\operatorname{Supp}^+_R(X, Y) \overset{\mathrm{def}}{=} \{\mathfrak{p} \in \operatorname{Proj} R^{\mathrm{ev}} \mid \operatorname{Hom}^*_{\mathcal{T}}(X, Y)_{\mathfrak{p}} \neq 0\}.$$

In the following theorem, we summarize some of the standard elementary properties of support sets (cf. [1]).

Theorem 1 (Properties of support)**.**

(1) $\operatorname{Supp}_R^+(X,Y) = \operatorname{Supp}_R^+ \operatorname{Hom}_{\mathcal{T}}^{\geq n}(X,Y)$ *for all* $n \in \mathbb{Z}$.

(2) *If* $\operatorname{Hom}_{\mathcal{T}}^{\geq n}(X,Y)$ *is a finitely generated* R*-module for some* n*, then*

$$\operatorname{Supp}_R^+(X,Y) = \{\mathfrak{p} \in \operatorname{Proj} R^{\mathrm{ev}} \mid \operatorname{Ann}_{R^{\mathrm{ev}}}\Big(\operatorname{Hom}_{\mathcal{T}}^{\geq n}(X,Y)\Big) \subseteq \mathfrak{p}\}.$$

In particular, if $\operatorname{Hom}_{\mathcal{T}}^*(X,Y) \in \operatorname{Noeth} R$*, then* $\operatorname{Supp}_R^+(X,Y)$ *is a closed set in* $\operatorname{Proj} R^{\mathrm{ev}}$.

(3) *If* $\operatorname{Hom}_{\mathcal{T}}^*(X,Y) \in \operatorname{Noeth} R$*, then* $\operatorname{Supp}_R^+(X,Y)$ *is empty if and only if* $\operatorname{Hom}_{\mathcal{T}}^*(X,Y)$ *is eventually zero.*

(4) *Given a triangle*

$$Z' \to Z \to Z'' \to \Sigma Z'$$

in $\mathcal{T}$*, there are inclusions*

$$\begin{aligned} \operatorname{Supp}_R^+(X,Z) &\subseteq \operatorname{Supp}_R^+(X,Z') \cup \operatorname{Supp}_R^+(X,Z''), \\ \operatorname{Supp}_R^+(Z,Y) &\subseteq \operatorname{Supp}_R^+(Z',Y) \cup \operatorname{Supp}_R^+(Z'',Y). \end{aligned}$$

(5) *If* G *is an object in* $\mathcal{T}$ *with* $\operatorname{thick}_{\mathcal{T}}(G) = \mathcal{T}$*, then*

$$\operatorname{Supp}_R^+(X,G) = \operatorname{Supp}_R^+(X,X) = \operatorname{Supp}_R^+(G,X).$$

Properties (3) and (5) provide a criterion for a finite dimensional algebra to be Gorenstein. For such an algebra Λ with radical $\mathfrak{r}$, the thick subcategory of $D^b(\Lambda)$ generated by the stalk complex $\Lambda/\mathfrak{r}$ is the whole of $D^b(\Lambda)$.

Corollary 2. *Let* Λ *be a finite dimensional algebra with radical* $\mathfrak{r}$*. Suppose that* $\operatorname{Ext}_\Lambda^*(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \in \operatorname{Noeth} R$ *for some graded-commutative ring* R *acting centrally on* $D^b(\Lambda)$ (*for example* $R = \operatorname{HH}^*(\Lambda)$)*. Then for every finitely generated* Λ*-module* M*, the implications*

$$\operatorname{pd} M < \infty \quad \Leftrightarrow \quad \operatorname{Ext}_\Lambda^n(M,M) = 0 \text{ for } n \gg 0 \quad \Leftrightarrow \quad \operatorname{id} M < \infty$$

hold. In particular, Λ *is Gorenstein.*

REFERENCES

[1] L. Avramov, S. Iyengar, *Modules with prescribed cohomological support*, Ill. J. Math. **51** (2007), 1–20.

[2] D. J. Benson, S. Iyengar, H. Krause, *Local cohomology and support for triangulated categories*, Ann. Sci. École Norm. Sup. série 4 **41** (2008), 575–621.

[3] D. J. Benson, S. Iyengar, H. Krause, *Stratifying modular representations of finite groups*, preprint 2008.

[4] P. A. Bergh, S. Iyengar, H. Krause, S. Oppermann, *Dimensions of triangulated categories via Koszul objects*, preprint 2008.

[5] R.-O. Buchweitz, H. Flenner, *Global Hochschild (co-)homology of singular spaces*, Adv. Math. **217** (2008), no. 1, 205–242.

[6] N. Yoneda, *Note on products in* Ext, Proc. Amer. Math. Soc. **9** (1958), 873–875.