ON GROWTH IN TOTALLY ACYCLIC MINIMAL COMPLEXES

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Abstract. Given a commutative Noetherian local ring, we provide a criterion under which a totally acyclic minimal complex of free modules has symmetric growth. In particular, we prove that whenever an image in the complex has finite complete intersection dimension, then the complex has symmetric polynomial growth.

1. Introduction

Given a commutative Noetherian local ring A, when does a totally acyclic minimal complex of free modules have symmetric growth? In other words, given such a complex, does the left growth of the ranks of its free modules equal the right growth? For M the image of a differential in such a complex, the left growth may be measured by the complexity of M, whereas the right growth may be measured by the complexity of its dual $M^* = \operatorname{Hom}_A(M, A)$. In this paper we study symmetric growth in the sense that both these invariants are finite and equal to one another:

$$\operatorname{cx}_A M = \operatorname{cx}_A M^*.$$

Avramov and Buchweitz show in [AvB] that this is always the case for totally acyclic minimal complexes of free modules over local complete intersections. However, Jorgensen and Şega showed in [JoŞ] that it does not hold for a local ring in general, even when the ring is Gorenstein. In fact, they constructed such a ring and a totally acyclic minimal complex whose left growth is exponential and right growth is constant. (The characteristics of growth in the dual complex are thus reversed.)

In this paper, we give a criterion under which symmetric growth of totally acyclic minimal complexes holds. This criterion is given in terms of the cohomology of the image of a given differential in the complex. Namely, we show that if the cohomology is finitely generated with respect to a ring acting centrally on the derived category, and the ring action commutes with dualization, then the complex has symmetric polynomial growth. As a corollary of our main theorem, we prove that whenever an image in the complex has finite complete intersection dimension, then the complex has symmetric polynomial growth. In other words, we show that the complexity of a module of complete intersection dimension zero equals that of its dual. Since all images in a totally acyclic minimal complex have complete intersection dimension zero when the ring is a local complete intersection, we recover the result of Avramov and Buchweitz cited above.

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2. Symmetric growth

Fix a local (meaning commutative Noetherian local) ring $(A, \mathfrak{m}, k)$ and a finitely generated A-module M. All modules we encounter are assumed to be finitely generated. We denote by M^* the A-dual of M, that is, the A-module $\operatorname{Hom}_A(M, A)$. If the canonical homomorphism $M \to M^{**}$ is an isomorphism, then M is said to be *reflexive.* Furthermore, if M is reflexive, and

$$\operatorname{Ext}_A^n(M, A) = 0 = \operatorname{Ext}_A^n(M^*, A)$$

for $n \geq 1$, then M is a module of *Gorenstein dimension zero* (alternatively, M is said to be *totally reflexive*). We shall write "G-dimension" instead of "Gorenstein dimension."

Suppose now that M has G-dimension zero, and fix free resolutions

$$\cdots \to C_2 \to C_1 \to C_0 \to M \to 0$$
$$\cdots \to C_{-3} \to C_{-2} \to C_{-1} \to M^* \to 0$$

of M and M^*, respectively. Dualizing the latter resolution, we obtain a complex

$$0 \to M \to C_{-1} \to C_{-2} \to C_{-3} \to \cdots$$

which is exact since $\operatorname{Ext}_A^n(M^*, A) = 0$ for $n \geq 1$. Splicing this sequence with the free resolution of M, we obtain a doubly infinite exact sequence

$$C\colon\ \cdots \to C_2 \to C_1 \to C_0 \xrightarrow{d_0} C_{-1} \to C_{-2} \to \cdots$$

of free modules, in which $M \cong \operatorname{Im} d_0$. The "left part" of the dualized complex $\operatorname{Hom}_A(C, A)$ is exact, since it is just the free resolution of M^*. Moreover, the "right part" is also exact, since $\operatorname{Ext}_A^n(M, A) = 0$ for $n \geq 1$. Consequently, the complex $\operatorname{Hom}_A(C, A)$ is exact; thus C is *totally acyclic.*

Conversely, if C is a totally acyclic complex of free modules, then the image of any of its differentials d_i has G-dimension zero. Given such a complex C, we denote by M_C the image of the zeroth differential, that is $M_C = \operatorname{Im} d_0$.

When we start with minimal free resolutions of M and M^*, we end up with an *almost minimal* totally acyclic complex C, meaning that the image of the differential $C_{n+1} \to C_n$ is contained in $\mathfrak{m} C_n$ for all $n \neq -1$. The complex C will be *minimal* if in addition the image of d_0 lies in $\mathfrak{m} C_{-1}$, and this happens precisely when M has no nonzero free summands.

Recall that if X is an A-module with a minimal free resolution

$$\cdots \to A^{\beta_2(X)} \to A^{\beta_1(X)} \to A^{\beta_0(X)} \to X \to 0,$$

then the rank $\beta_n(X)$ is the nth *Betti number* of X. This equals the dimension of the k-vector space $\operatorname{Ext}_A^n(X, k)$. We shall denote the Poincaré series $\sum_{n=0}^{\infty} \beta_n(X) t^n$ of the Betti sequence by $P_A(X, t)$. The *complexity* of X, denoted $\operatorname{cx}_A X$, is defined as

$$\operatorname{cx}_A X \overset{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(X) \leq a n^{t-1} \text{ for } n \gg 0\}.$$

In other words, the complexity of X measures the polynomial rate of growth of the *Betti numbers* of X. Thus, a totally acyclic minimal complex of free modules C has *symmetric polynomial growth* if $\operatorname{cx}_A M_C = \operatorname{cx}_A M_C^* < \infty$.

The aim of this paper is to give sufficient conditions for a totally acyclic minimal complex of free modules to have symmetric polynomial growth. In other words, given such a complex C, we give a criterion for when $\operatorname{cx}_A M_C$ and $\operatorname{cx}_A M_C^*$

are equal. Namely, we show that this happens whenever the cohomology of M_C behaves well with respect to dualization and is "finitely generated." To explain these notions, consider the bounded derived category $D^b(A)$ of finitely generated A-modules. This is a triangulated category, whose suspension functor Σ is the left shift of a complex. Given complexes X, Y and an integer n, we denote the graded A-module $\operatorname{Hom}_{D^b(A)}(X, \Sigma^n Y)$ by $\operatorname{Ext}^n_A(X,Y)$; for modules, this is the usual Ext. We denote by $\operatorname{Ext}_A(X,Y)$ the graded module $\oplus_{n\in\mathbb{Z}} \operatorname{Ext}^n_A(X,Y)$, and if n_0 is an integer then we set $\operatorname{Ext}^{\geq n_0}_A(X,Y) = \oplus_{n\geq n_0} \operatorname{Ext}^n_A(X,Y)$. The *graded center* of $D^b(A)$, denoted $Z(D^b(A))$, is a graded ring $Z(D^b(A)) = \oplus_{n\in\mathbb{Z}} Z^n(D^b(A))$, whose degree n component $Z^n(D^b(A))$ is the set of natural transformations $\operatorname{Id} \xrightarrow{f} \Sigma^n$ satisfying $f_{\Sigma X} = (-1)^n \Sigma f_X$ on the level of objects. For details and properties of the graded center, see [BuF].

Now let $H = \oplus_{n\geq 0} H_n$ be a positively graded ring which is graded-commutative, that is, $\eta\theta = (-1)^{|\eta||\theta|}\theta\eta$ for all homogeneous elements $\eta, \theta \in H$. We say that H *acts centrally* on $D^b(A)$ if there exists a homomorphism $H \to Z(D^b(A))$ of graded rings. In this case, for every complex $X \in D^b(A)$ there is a graded ring homomorphism

$$H \xrightarrow{\varphi_X} \operatorname{Ext}_A(X,X),$$

and for every complex Y and all homogeneous elements $\eta \in H, \theta \in \operatorname{Ext}_A(X,Y)$ the equality $\varphi_Y(\eta) \circ \theta = (-1)^{|\eta||\theta|}\theta \circ \varphi_X(\eta)$ holds. In other words, the left and right H-module structures on $\operatorname{Ext}_A(X,Y)$ coincide up to a sign.

Definition. We say that $\operatorname{Ext}_A(X,Y)$ is an *eventually Noetherian H-module of finite length*, and write $\operatorname{Ext}_A(X,Y) \in \operatorname{Noeth}^{\mathrm{fl}} H$, if the following holds: there is a number n_0 such that the H-module $\operatorname{Ext}^{\geq n_0}_A(X,Y)$ is Noetherian and the length $\ell_{H_0} \operatorname{Ext}^n_A(X,Y)$ of $\operatorname{Ext}^n_A(X,Y)$ as an H_0-module is finite for each $n \geq n_0$.

The following lemma shows that if the cohomology of a module is finitely generated in the sense of the above definition, then the complexity of the module is finite, and its Poincaré series is a rational function. This result is "folklore" in homological algebra, but we include it due to the lack of a precise reference.

Lemma 2.1. *Let $(A, \mathfrak{m}, k)$ be a local ring and M a finitely generated A-module. Suppose that $\operatorname{Ext}_A(M,k)$ belongs to $\operatorname{Noeth}^{\mathrm{fl}} H$ for some graded-commutative ring H acting centrally on $D^b(A)$. Then the complexity of M is finite and the Poincaré series $P_A(M,t)$ is rational. Moreover, the complexity of M equals the order of the pole of $P_A(M,t)$ at $t = 1$.*

Proof. Since the scalar action from H_0 on $\operatorname{Ext}^n_A(M,k)$ factors through $\operatorname{Hom}_A(k,k)$, the complexity of M is the rate of growth of the sequence $\{\ell_{H_0} \operatorname{Ext}^n_A(M,k)\}_{n=0}^{\infty}$. Therefore, by [BIKO, Lemma 2.6], the complexity of M is finite. Now let n_0 be an integer such that the H-module $\operatorname{Ext}^{\geq n_0}_A(M,k)$ is Noetherian and $\ell_{H_0}(\operatorname{Ext}^n_A(M,k)) < \infty$ for each $n \geq n_0$, and denote the ideal $\operatorname{Ann}_H \operatorname{Ext}^{\geq n_0}_A(M,k)$ in H by I. By [BIKO, Remark 2.1], the quotient ring H/I is Noetherian, and its degree zero part $(H/I)_0$ is Artin. The rationality of $P_A(M,t)$ now follows from the Hilbert-Serre Theorem (cf. [AtM, Theorem 11.1]). The last part is a standard result on Poincaré series (cf. [Ben, Proposition 5.3.2]). □

Let M be an A-module of G-dimension zero, C a totally acyclic complex of free modules with $M = M_C$, and θ an element of $\operatorname{Ext}^n_A(M,M)$ for some n. This element θ corresponds to a chain map $C \to \Sigma^n C$ (and equivalent elements in $\operatorname{Ext}_A^n(M,M)$ correspond to homotopic chain maps). Dualizing, we obtain a chain map $\Sigma^{-n}(C^*) = (\Sigma^n C)^* \to C^*$. Applying the shift functor Σ^n we now get a chain map $C^* \to \Sigma^n(C^*)$, and this corresponds to an element in $\operatorname{Ext}_A^n(M^*, M^*)$. One checks easily that this defines an anti-isomorphism

$$\mathcal{D} \colon \operatorname{Ext}_A(M,M) \to \operatorname{Ext}_A(M^*,M^*)$$

of graded rings.

Definition. Let M be an A-module of G-dimension zero. We say that the central ring action from H on $D^b(A)$ *commutes with dualization of* M, provided that the diagram

$$\begin{tikzcd}
& H \arrow[dl, "\varphi_M"'] \arrow[dr, "\varphi_{M^*}"] & \\
\operatorname{Ext}_A(M,M) \arrow[rr, "\mathcal{D}"] & & \operatorname{Ext}_A(M^*,M^*)
\end{tikzcd}$$

commutes, that is, $\mathcal{D}(\varphi_M(\eta)) = \varphi_{M^*}(\eta)$ for every homogeneous element $\eta \in H$.

Before proving the main theorem, recall that a full subcategory of $D^b(A)$ is *thick* if it is triangulated and closed under direct summands. The thick subcategory generated by a given object X, denoted $\operatorname{thick}_{D^b(A)}(X)$, is the intersection of all the thick subcategories containing it. Recall also that the notion of dualization and G-dimension makes perfectly sense in $D^b(A)$, and we refer to [Chr] for more details.

Having established all the necessary terminology and concepts, we now prove the main theorem. It shows that if a module has G-dimension zero, its cohomology is finitely generated, and the central ring action on the derived category commutes with dualization, then the complexity of the module equals that of its dual.

Theorem 2.2. *Let* $(A, \mathfrak{m}, k)$ *be a local ring, and* M *a finitely generated* A*-module of G-dimension zero. If* $\operatorname{Ext}_A(M \oplus M^*, k)$ *belongs to* $\operatorname{Noeth}^{\mathrm{fl}} H$ *for some graded-commutative ring* H *acting centrally on* $D^b(A)$*, and such that its action commutes with dualization for all objects in* $\operatorname{thick}_{D^b(A)}(M)$*, then* $\operatorname{cx}_A M = \operatorname{cx}_A M^*$*.*

Proof. We prove the result by induction on $\operatorname{cx}_A M$, which is finite by Lemma 2.1. If $\operatorname{cx}_A M = 0$, then M has finite projective dimension and is therefore free. But then M^* is also free, and the equality $\operatorname{cx}_A M = \operatorname{cx}_A M^*$ trivially holds.

Next, suppose that $\operatorname{cx}_A M$ is nonzero. By [BIKO, Lemma 2.5], there exists a homogeneous element $\eta \in H$, of positive degree, inducing injective maps

$$\begin{aligned} \operatorname{Ext}_A^n(M,k) &\xrightarrow{\cdot\varphi_M(\eta)} \operatorname{Ext}_A^{n+|\eta|}(M,k) \\ \operatorname{Ext}_A^n(M^*,k) &\xrightarrow{\cdot\varphi_{M^*}(\eta)} \operatorname{Ext}_A^{n+|\eta|}(M^*,k) \end{aligned}$$

for $n \gg 0$. Choose a short exact sequence

$$(\dagger) \qquad 0 \to M \to K \to \Omega_A^{|\eta|-1}(M) \to 0$$

representing the element $\varphi_M(\eta)$ in $\operatorname{Ext}_A^{|\eta|}(M,M)$. Since the module $\Omega_A^{|\eta|-1}(M)$ has G-dimension zero, when we dualize this sequence we obtain a short exact sequence

$$(\ddagger) \qquad 0 \to \Omega_A^{|\eta|-1}(M)^* \to K^* \to M^* \to 0.$$

Since both the end term modules in (†) have G-dimension zero, so does the module K by [Chr, Lemma 1.1.10]. Moreover, since the H-module $\operatorname{Ext}_A(X,k)$ belongs to $\operatorname{Noeth}^{\mathrm{fl}} H$ whenever X is one of the end term modules in (†) and (‡), so does $\operatorname{Ext}_A(K \oplus K^*, k)$.

The sequence (†) induces a long exact sequence

$$\cdots \to \operatorname{Ext}^n_A(K,k) \to \operatorname{Ext}^n_A(M,k) \xrightarrow{\partial_n} \operatorname{Ext}^{n+|\eta|}_A(M,k) \to \operatorname{Ext}^{n+1}_A(K,k) \to \cdots$$

in cohomology. The connecting homomorphism ∂_n is just scalar multiplication with $\varphi_M(\eta)$ (cf. the proof of [AGP, Theorem 7.8]), and is therefore injective for $n \gg 0$. The sequence (‡) also induces a cohomological long exact sequence, which takes the form

$$\cdots \to \operatorname{Ext}^{n-1}_A(K^*,k) \to \operatorname{Ext}^{n-|\eta|}_A(M^*,k) \xrightarrow{\partial'_n} \operatorname{Ext}^n_A(M^*,k) \to \operatorname{Ext}^n_A(K^*,k) \to \cdots$$

for $n > |\eta|$. Moreover, for such n, the connecting homomorphism ∂'_n is scalar multiplication with $\mathcal{D}(\varphi_M(\eta))$. By assumption the central ring action from H on $D^b(A)$ commutes with dualization of M, that is, $\mathcal{D}(\varphi_M(\eta)) = \varphi_{M^*}(\eta)$. Therefore the connecting homomorphism ∂'_n is also injective for $n \gg 0$.

Choose a number $n_0 \geq |\eta|$ with the property that both ∂_n and ∂'_n are injective for $n \geq n_0$. Then the sequences

$$0 \to \operatorname{Ext}^n_A(M,k) \to \operatorname{Ext}^{n+|\eta|}_A(M,k) \to \operatorname{Ext}^{n+1}_A(K,k) \to 0$$
$$0 \to \operatorname{Ext}^{n-|\eta|}_A(M^*,k) \to \operatorname{Ext}^n_A(M^*,k) \to \operatorname{Ext}^n_A(K^*,k) \to 0$$

are exact for $n \geq n_0$, giving equalities

$$\begin{aligned} \beta_{n+1}(K) &= \beta_{n+|\eta|}(M) - \beta_n(M) \\ \beta_n(K^*) &= \beta_n(M^*) - \beta_{n-|\eta|}(M^*) \end{aligned}$$

of Betti numbers. Computing Poincaré series, we obtain

$$\begin{aligned} P_A(K,t) &= \frac{(1-t^{|\eta|})P_A(M,t)}{t^{|\eta|-1}} + f(t) \\ P_A(K^*,t) &= (t^{|\eta|}-1)P_A(M^*,t) + g(t) \end{aligned}$$

for some polynomials $f(t), g(t) \in \mathbb{Z}[t]$. Consequently, the order of the pole of $P_A(K,t)$ at $t = 1$ is one less than that of $P_A(M,t)$, whereas the pole of $P_A(K^*,t)$ is one less than that of $P_A(M^*,t)$. Therefore, by Lemma 2.1, the complexity of K is $\operatorname{cx}_A M - 1$. As an object of $D^b(A)$, the module K belongs to $\operatorname{thick}_{D^b(A)}(M)$, and so by induction we obtain

$$\operatorname{cx}_A M = \operatorname{cx}_A K + 1 = \operatorname{cx}_A K^* + 1 = \operatorname{cx}_A M^*.$$

This completes the proof. □

As an immediate corollary, we provide the criterion under which a totally acyclic minimal complex of free modules has symmetric growth, a criterion given in terms of the cohomology of the image of any of the differentials. Namely, if the cohomology is finitely generated with respect to a ring acting centrally on the derived category, and the ring action commutes with dualization, then the complex has symmetric polynomial growth.

Corollary 2.3. *Let $(A, \mathfrak{m}, k)$ be a local ring, and C a totally acyclic minimal complex of finitely generated free A-modules. Suppose that $\operatorname{Ext}_A(M_C \oplus M_C^*, k)$ belongs to $\operatorname{Noeth}^{\mathrm{fl}} H$ for some graded-commutative ring H acting centrally on $D^b(A)$, and such that its action commutes with dualization for all objects in $\operatorname{thick}_{D^b(A)}(M_C)$. Then C has symmetric growth.*

3. Finite complete intersection dimension

Assume that $A = B/(x_1, \dots, x_c)$, where $(B, \mathfrak{n})$ is a local ring and $x_1, \dots, x_c$ is a B-regular sequence contained in $\mathfrak{n}$. Then there exists a polynomial ring

$$H = A[\chi_1, \dots, \chi_c]$$

acting centrally on $D^b(A)$, with each *cohomology operator* χ_i of degree two. For purposes below, we recall the definition of the elements $\varphi_M(\chi_i) \in \operatorname{Ext}^2_A(M, M)$, and their action on $\operatorname{Ext}_A(M, N)$ and $\operatorname{Ext}_A(N, M)$. There are actually several definitions for these elements, but they all agree up to sign; see [AS]. The one we give is from [Eis].

Let (C, d) be a complex of free A-modules. We lift C to a sequence of maps $(\widetilde{C}, \widetilde{d})$ of free B-modules. Since $\widetilde{d}^2 \equiv 0$ modulo $(x_1, \dots, x_c)$ we can decompose $\widetilde{d}^2$ as

$$\widetilde{d}^2 = \sum_{i=1}^{c} x_i \widetilde{t}_i$$

for some family $(\widetilde{t}_i)_{i=1}^c$ of degree -2 endomorphisms of the graded B-module $\widetilde{C}$. Then $t_i = \widetilde{t}_i \otimes_B A$ become degree -2 chain maps on the complex C which are well-defined and commute up to homotopy (see [Eis]). The chain maps t_i on a free resolution C of M then define the elements $\varphi_M(\chi_i) \in \operatorname{Ext}^2_A(M, M)$. The action of $\varphi_M(\chi_i)$ on $\operatorname{Ext}_A(M, N)$ and $\operatorname{Ext}_A(N, M)$ is thus determined by composition of chain maps. If $M = M_C$ for a totally acyclic complex C of free A-modules, the element $\varphi_M(\chi_i) \in \operatorname{Ext}^2_A(M, M)$ is determined by the chain map $t_i \colon C \to \Sigma^2 C$.

Recall that a *quasi-deformation* of local rings is a diagram

$$A \to A' \leftarrow B$$

of local homomorphisms satisfying the following: the map $A \to A'$ is flat, and $A' \leftarrow B$ is surjective with kernel generated by a B-regular sequence contained in the maximal ideal of B. If M is a finitely generated module over a local ring A, then its *complete intersection dimension*, denoted $\operatorname{CI-dim}_A M$, is defined as

$$\operatorname{CI-dim}_A M \overset{\text{def}}{=} \inf\{\operatorname{pd}_B(M \otimes_A A') - \operatorname{pd}_B A' \mid A \to A' \leftarrow B \text{ is a quasi-deformation}\}.$$

This was introduced in [AGP], where it was shown that this invariant dominates the G-dimension and satisfies the "Auslander-Buchsbaum formula". Namely, if M has finite CI-dimension, then it also has finite G-dimension and

$$\operatorname{CI-dim}_A M = \operatorname{G-dim}_A M = \operatorname{depth} A - \operatorname{depth} M.$$

Lemma 3.1. *Let $A = B/(x_1, \dots, x_c)$, where $(B, \mathfrak{n}, k)$ is a local ring and $x_1, \dots, x_c$ is a B-regular sequence contained in $\mathfrak{n}$. Suppose that M is an A-module with $\operatorname{depth}_A M = \operatorname{depth}_A A$, and $\operatorname{pd}_B M < \infty$. Then the central ring action of the polynomial ring of cohomology operators $H = A[\chi_1, \dots, \chi_c]$ on $D^b(A)$ commutes with dualization of M.*

Proof. The hypotheses show that the CI-dimension of M is zero, and therefore M has G-dimension zero. Let C be a totally acyclic complex of free modules with $M = M_C$. Choose $\chi = \chi_i \in H$. It suffices to prove that $\mathcal{D}(\varphi_M(\chi)) = \varphi_{M^*}(\chi)$. The element $\varphi_M(\chi) \in \operatorname{Ext}_A(M, M)$ is determined by the chain map $t \colon C \to \Sigma^2 C$, as described above. Therefore $\mathcal{D}(\varphi_M(\chi))$ corresponds to the chain map $\Sigma^2 t^* \colon C^* \to \Sigma^2 C^*$. On the other hand, if one uses the lifting $\operatorname{Hom}_B(\widetilde{C}, B)$ of C^* with lifted maps

$\big(\widetilde{d}\,\big)^* = \operatorname{Hom}_A(\widetilde{d}, B)$, and the factorization of $\big(\big(\widetilde{d}\,\big)^*\big)^2$ dual to that of $\widetilde{d}^2$ we see that the chain map $\Sigma^2\, t^* : C^* \to \Sigma^2\, C^*$ defines the element $\varphi_{M^*}(\chi) \in \operatorname{Ext}_A(M^*, M^*)$, and this is what we wanted to show. ☐

Remark. As with the notion of G-dimension, the notion of complete intersection dimension makes sense for complexes in the bounded derived category (cf. [S-W]). Not surprisingly, Lemma 3.1 holds for complexes of finite CI-dimension. Moreover, given such a complex X in $D^b(A)$, every complex in $\operatorname{thick}_{D^b(A)}(X)$ also has finite CI-dimension, over the same quasi-deformation as X (cf. [Ber]).

As a consequence of Lemma 3.1 and the above remark, we obtain the following result, which is an application of Theorem 2.2 in the case when the module has CI-dimension zero.

Theorem 3.2. *Let $(A, \mathfrak{m}, k)$ be a local ring, and M a finitely generated A-module with* $\text{CI-dim}_A M = 0$ *Then* $\operatorname{cx}_A M = \operatorname{cx}_A M^*$.

Proof. By [BeJ, Lemma 3.5] and its proof, the CI-dimension of M^* is also zero, and there exists a quasi deformation

$$A \to A' \leftarrow B$$

such that $\operatorname{pd}_B((M \oplus M^*) \otimes_A A') < \infty$. Moreover, from the isomorphism

$$\operatorname{Hom}_A(M, A) \otimes_A A' \cong \operatorname{Hom}_{A'}(M \otimes_A A', A')$$

we see that the complexity of the A-dual M^* of M equals that of the A'-module $\operatorname{Hom}_{A'}(M \otimes_A A', A')$, which is the A'-dual of $M \otimes_A A'$. Finally, the projective dimension of the B-module $(M \otimes_A A') \oplus \operatorname{Hom}_{A'}(M \otimes_A A', A')$ is finite, and the equality

$$\operatorname{depth}_A A - \operatorname{depth}_A M = \operatorname{depth}_{A'} A' - \operatorname{depth}_{A'}(M \otimes_A A')$$

holds. Changing notation, we can therefore assume that $A = A'$, and so $A = B/(x_1, \dots, x_c)$, where B is a local ring and $x_1 \dots, x_c$ is a B-regular sequence contained in the maximal ideal of B. Moreover, the projective dimension of the B-module $M \oplus M^*$ is finite, the G-dimension of M is zero, and $\operatorname{depth}_A M = \operatorname{depth}_A A$. By Lemma 3.1 and the subsequent remark, the central ring action of the polynomial ring of cohomology operators $H = A[\chi_1, \dots, \chi_c]$ on $D^b(A)$ commutes with dualization for every object in $\operatorname{thick}_{D^b(A)}(M)$. Therefore, by Theorem 2.2, it suffices to show that $\operatorname{Ext}_A(M \oplus M^*, k)$ belongs to $\operatorname{Noeth}^{\text{fl}} H$, but this is implied by the main result of [Gul] (see also [Eis] and [Avr]). ☐

Corollary 3.3. *Let $(A, \mathfrak{m}, k)$ be a local ring, and C a totally acyclic minimal complex of finitely generated free A-modules. If M_C has finite CI-dimension, then C has symmetric growth.*

Proof. Since M_C is an infinite syzygy, its CI-dimension is zero by [AGP, Lemma 1.9]. ☐

When A is a complete intersection, then every finitely generated A-module has finite CI-dimension. Therefore, in this case, every totally acyclic minimal complex of free modules has symmetric growth. This follows already from [AvB, Theorem 5.6], but because of our alternative proof, we include it here as a corollary.

Corollary 3.4. *Let $(A, \mathfrak{m}, k)$ be a local complete intersection. Then every totally acyclic minimal complex of finitely generated free A-modules has symmetric growth.*

We end this paper with a question on central ring actions and dualizations.

Question 3.5. Do there exist graded commutative rings H acting centrally on $D^b(A)$ such that the action fails to commute with dualization of a finitely generated A-module M?

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