HOCHSCHILD HOMOLOGY AND SPLIT PAIRS

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Abstract. We study the Hochschild homology of algebras related via split pairs, and apply this to fibre products, trivial extensions, monomial algebras, graded-commutative algebras and quantum complete intersections. In particular, we compute lower bounds for the dimensions of both the Hochschild homology and cohomology groups of quantum complete intersections.

1. Introduction

The Hochschild homology of a finite dimensional algebra is an invariant about which little is known. One of the main unsolved problems is whether an algebra of infinite global dimension can have finite total Hochschild homology. This started as a question in cohomology. In [Hap], Happel noted that if a finite dimensional algebra over an algebraically closed field is of finite global dimension, then all its higher Hochschild cohomology groups vanish. He then remarked that "the converse seems to be not known". Thus, the question of whether all the higher Hochschild cohomology groups can vanish for a finite dimensional algebra (not necessarily over an algebraically closed ground field) of infinite global dimension became known as "Happel's question". As shown in [AvI], the answer is yes when the algebra is commutative. However, it was shown in [BGMS] that this does not hold in general.

The homology version of Happel's question is still open: if all the higher Hochschild homology groups of a finite dimensional algebra vanish, then is the algebra of finite global dimension? As in the cohomology case, the answer is yes for commutative algebras (cf. [AV-P]). It was conjectured by Han in [Han] to hold for all finite dimensional algebras, and is therefore known as "Han's conjecture". As mentioned, this is one of the main unsolved problems concerning Hochschild homology.

In this paper, we study the functorial behavior of Hochschild homology. In particular, we study the Hochschild homology of algebras which are related via split pairs. Given an algebra together with a split pair subalgebra, the homology of this subalgebra is a subspace of the homology of the larger algebra. Given several such subalgebras, one may ask: do the corresponding homology submodules intersect nontrivially? We prove that, in many cases, they do not. Consequently, the direct sum of the homology of the subalgebras is a submodule of the homology of the larger algebra. Thus, we obtain a lower bound for the dimension of the homology groups of the algebra. We apply this to fibre products of algebras, trivial extensions, monomial algebras, path algebras of quivers containing loops, graded-commutative algebras, and to quantum complete intersections. For the latter, we compute lower bounds for the dimensions of both the Hochschild homology and cohomology groups.

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2. Hochschild homology

Let k be a commutative ring, and let Λ be a k-algebra. Thus, there exists a nonzero ring homomorphism $k \to \Lambda$, whose image is contained in the center of Λ. For now, we do not assume that Λ is finitely generated as a k-module. Let B be a Λ-Λ-bimodule. For each $n \geq 1$, consider the map

$$\begin{array}{rcl} B \otimes \Lambda^{\otimes n} & \xrightarrow{d_n} & B \otimes \Lambda^{\otimes (n-1)} \\ b \otimes \lambda_1 \otimes \cdots \otimes \lambda_n & \mapsto & b \cdot \lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_n \\ & & + \sum_{i=1}^{n-1} (-1)^i b \otimes \lambda_1 \otimes \cdots \otimes \lambda_i \lambda_{i+1} \otimes \cdots \otimes \lambda_n \\ & & + (-1)^n \lambda_n \cdot b \otimes \lambda_1 \otimes \cdots \otimes \lambda_{n-1}, \end{array}$$

where unadorned tensor products are tensor products over the ground ring k. Then $d_n d_{n+1} = 0$ (cf. [CaE, IX.6]), and so we may form a complex

$$\cdots \to B \otimes \Lambda^{\otimes 3} \xrightarrow{d_3} B \otimes \Lambda^{\otimes 2} \xrightarrow{d_2} B \otimes \Lambda \xrightarrow{d_1} B \to 0$$

in which B is in degree 0. The nth homology group of this complex, denoted $\operatorname{HH}_n(\Lambda, B)$, is the nth *Hochschild homology* group of Λ with coefficients in B. This is a k-module, in general not finitely generated. Of particular interest is the special case $B = \Lambda$; we write $\operatorname{HH}_n(\Lambda)$ instead of $\operatorname{HH}_n(\Lambda, \Lambda)$, and in this case we refer to the above complex as the *Hochschild complex* of Λ.

In this paper, we shall study the ordinary Hochschild homology groups $\operatorname{HH}_n(\Lambda)$, as well as homology groups $\operatorname{HH}_n(\Lambda, {}_{\psi}\Lambda_1)$, with coefficients in twisted bimodules of the form ${}_{\psi}\Lambda_1$, where ψ is an algebra automorphism of Λ. The bimodule structure of ${}_{\psi}\Lambda_1$ is given by letting Λ act from the left via ψ, i.e.

$$\lambda_1 \cdot \lambda \cdot \lambda_2 \stackrel{\text{def}}{=} \psi(\lambda_1)\lambda\lambda_2$$

for $\lambda \in {}_{\psi}\Lambda_1$ and $\lambda_1, \lambda_2 \in \Lambda$. As the following lemma shows, Hochschild homology with coefficients in twisted bimodules is functorial when we only consider maps commuting with the automorphisms.

Lemma 2.1. *Let k be a commutative ring, and let Γ and Λ be k-algebras. Furthermore, let $\Gamma \xrightarrow{\phi} \Gamma$ and $\Lambda \xrightarrow{\psi} \Lambda$ be automorphisms. Then for any $n \geq 0$, an algebra homomorphism $\Gamma \xrightarrow{f} \Lambda$ such that $f\phi = \psi f$ induces a k-homomorphism $\operatorname{HH}_n(\Gamma, {}_{\phi}\Gamma_1) \xrightarrow{f^{\otimes(n+1)}} \operatorname{HH}_n(\Lambda, {}_{\psi}\Lambda_1)$ given by*

$$\sum_i \gamma^i \otimes \gamma_1^i \otimes \cdots \otimes \gamma_n^i + \operatorname{Im} d_{n+1} \mapsto \sum_i f(\gamma^i) \otimes f(\gamma_1^i) \otimes \cdots \otimes f(\gamma_n^i) + \operatorname{Im} d_{n+1}.$$

Proof. For every n the map f induces a map ${}_{\phi}\Gamma_1 \otimes \Gamma^{\otimes n} \xrightarrow{f^{\otimes(n+1)}} {}_{\psi}\Lambda_1 \otimes \Lambda^{\otimes n}$ of k-modules, by applying f to each tensor component. Let $n \geq 1$, and consider the diagram

$$\begin{array}{ccc} {}_{\phi}\Gamma_1 \otimes \Gamma^{\otimes n} & \xrightarrow{d_n} & {}_{\phi}\Gamma_1 \otimes \Gamma^{\otimes (n-1)} \\ \downarrow{\scriptstyle f^{\otimes(n+1)}} & & \downarrow{\scriptstyle f^{\otimes n}} \\ {}_{\psi}\Lambda_1 \otimes \Lambda^{\otimes n} & \xrightarrow{d_n} & {}_{\psi}\Lambda_1 \otimes \Lambda^{\otimes (n-1)} \end{array}$$

Given a generator $w = \gamma \otimes \gamma_1 \otimes \cdots \otimes \gamma_n$ in ${}_{\phi}\Gamma_1 \otimes \Gamma^{\otimes n}$, direct computation gives

$$\begin{aligned} f^{\otimes n} d_n(w) &= f(\gamma\gamma_1) \otimes f(\gamma_2) \otimes \cdots \otimes f(\gamma_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i f(\gamma) \otimes f(\gamma_1) \otimes \cdots \otimes f(\gamma_i \gamma_{i+1}) \otimes \cdots \otimes f(\gamma_n) \\ &\quad + (-1)^n f(\phi(\gamma_n)\gamma) \otimes f(\gamma_2) \otimes \cdots \otimes f(\gamma_{n-1}), \end{aligned}$$

whereas

$$\begin{aligned} d_n f^{\otimes (n+1)}(w) &= f(\gamma) f(\gamma_1) \otimes f(\gamma_2) \otimes \cdots \otimes f(\gamma_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i f(\gamma) \otimes f(\gamma_1) \otimes \cdots \otimes f(\gamma_i) f(\gamma_{i+1}) \otimes \cdots \otimes f(\gamma_n) \\ &\quad + (-1)^n \psi f(\gamma_n) f(\gamma) \otimes f(\gamma_2) \otimes \cdots \otimes f(\gamma_{n-1}), \end{aligned}$$

Since f is a ring homomorphism and $f\phi = \psi f$, we see that $f^{\otimes n} d_n(w)$ equals $d_n f^{\otimes (n+1)}(w)$, that is, the above diagram is commutative. Thus we obtain a map

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & {}_{\phi}\Gamma_1 \otimes \Gamma^{\otimes 3} & \xrightarrow{d_3} & {}_{\phi}\Gamma_1 \otimes \Gamma^{\otimes 2} & \xrightarrow{d_2} & {}_{\phi}\Gamma_1 \otimes \Gamma & \xrightarrow{d_1} & {}_{\phi}\Gamma_1 & \longrightarrow & 0 \\ & \downarrow f^{\otimes 4} & & \downarrow f^{\otimes 3} & & \downarrow f^{\otimes 2} & & \downarrow f & & \\ \cdots \longrightarrow & {}_{\psi}\Lambda_1 \otimes \Lambda^{\otimes 3} & \xrightarrow{d_3} & {}_{\psi}\Lambda_1 \otimes \Lambda^{\otimes 2} & \xrightarrow{d_2} & {}_{\psi}\Lambda_1 \otimes \Lambda & \xrightarrow{d_1} & {}_{\psi}\Lambda_1 & \longrightarrow & 0 \end{array}$$

of complexes, and the proof is complete. □

In particular, Hochschild homology $\mathrm{HH}_*(\Lambda)$ (with coefficients in the algebra itself) is functorial in the ordinary sense.

3. Split pairs

Given two objects X, Y in a category, we say that the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

is a *split pair* if the composition gf is the identity morphism on X. Now let k be a commutative ring, and let Γ and Λ be k-algebras, that is, there are ring homomorphisms from k into the centers of the two algebras. Furthermore, let

$$\Gamma \xrightarrow{j} \Lambda \xrightarrow{\pi} \Gamma$$

be a split pair. Then $\pi(\lambda - j\pi(\lambda)) = 0$ for all $\lambda \in \Lambda$, and consequently $\lambda \in \operatorname{Im} j + \operatorname{Ker} \pi$. Since $\operatorname{Im} j \cap \operatorname{Ker} \pi = 0$, it follows that $\Lambda = \operatorname{Im} j \oplus \operatorname{Ker} \pi$ as a k-module. Let $\Lambda \xrightarrow{\psi} \Lambda$ be an automorphism under which $\operatorname{Im} j$ and $\operatorname{Ker} \pi$ are both invariant, i.e. $\psi(\operatorname{Im} j) \subseteq \operatorname{Im} j$ and $\psi(\operatorname{Ker} \pi) \subseteq \operatorname{Ker} \pi$. Then since ψ is an automorphism and $\operatorname{Im} j \cap \operatorname{Ker} \pi = 0$, the inclusions are actually equalities, i.e. $\psi(\operatorname{Im} j) = \operatorname{Im} j$ and $\psi(\operatorname{Ker} \pi) = \operatorname{Ker} \pi$. Now define an algebra homomorphism

$$\begin{array}{ccc} \Gamma & \xrightarrow{\phi} & \Gamma \\ \gamma & \mapsto & \pi\psi j(\gamma). \end{array}$$

This is easily seen to be an automorphism of Γ. The following result shows that the Hochschild homology of Γ with coefficients in ${}_{\phi}\Gamma_1$ is a direct summand of the Hochschild homology of Λ with coefficients in ${}_{\psi}\Lambda_1$.

Proposition 3.1. *Let k be a commutative ring, and let $\Gamma \xrightarrow{j} \Lambda \xrightarrow{\pi} \Gamma$ be a split pair of k-algebras. Furthermore, let $\Lambda \xrightarrow{\psi} \Lambda$ be an automorphism under which* $\operatorname{Im} j$ *and*

$\operatorname{Ker}\pi$ are both invariant, and define $\Gamma \xrightarrow{\phi} \Gamma$ by $\phi = \pi\psi j$. Then for every $n \geq 0$, the diagram

$$\operatorname{HH}_n(\Gamma, {}_{\phi}\Gamma_1) \xrightarrow{j^{\otimes(n+1)}} \operatorname{HH}_n(\Lambda, {}_{\psi}\Lambda_1) \xrightarrow{\pi^{\otimes(n+1)}} \operatorname{HH}_n(\Gamma, {}_{\phi}\Gamma_1)$$

is a split pair of k-modules. In particular, the k-module $\operatorname{HH}_n(\Gamma, {}_{\phi}\Gamma_1)$ is isomorphic to a direct summand of $\operatorname{HH}_n(\Lambda, {}_{\psi}\Lambda_1)$.

Proof. Let γ be any element of Γ. Since $\operatorname{Im} j$ is invariant under ψ, the element $\psi j(\gamma)$ is of the form $j(\gamma')$ for some element $\gamma' \in \Gamma$. This gives

$$j\phi(\gamma) = j\pi\psi j(\gamma) = j\pi j(\gamma') = j(\gamma') = \psi j(\gamma),$$

showing $j\phi = \psi j$.

Now let λ be any element of Λ. Since $\Lambda = \operatorname{Im} j \oplus \operatorname{Ker}\pi$, we can write $\lambda = j(\gamma) + w$, where $\gamma \in \Gamma$ and $w \in \operatorname{Ker}\pi$. Moreover, since $\operatorname{Ker}\pi$ is invariant under ψ, the element $\psi(w)$ is mapped to zero under π. Therefore

$$\pi\psi(\lambda) = \pi\psi(j(\gamma) + w) = \pi\psi j(\gamma) = \phi(\gamma) = \phi\pi(\lambda),$$

showing $\pi\psi = \phi\pi$.

By Lemma 2.1, for every $n \geq 0$ the maps j and π induce k-homomorphisms

$$\operatorname{HH}_n(\Gamma, {}_{\phi}\Gamma_1) \xrightarrow{j^{\otimes(n+1)}} \operatorname{HH}_n(\Lambda, {}_{\psi}\Lambda_1) \xrightarrow{\pi^{\otimes(n+1)}} \operatorname{HH}_n(\Gamma, {}_{\phi}\Gamma_1).$$

This diagram is obviously a split pair of k-modules. □

Thus, in the setting described in the proposition, we may view $\operatorname{HH}_n(\Gamma, {}_{\phi}\Gamma_1)$ as a direct summand of $\operatorname{HH}_n(\Lambda, {}_{\psi}\Lambda_1)$.

Remarks 3.2. (i) When ψ (and hence also ϕ) is the identity automorphism, then $\operatorname{Im} j$ and $\operatorname{Ker}\pi$ are of course automatically invariant. That is, if $\Gamma \xrightarrow{j} \Lambda \xrightarrow{\pi} \Gamma$ is any split pair of k-algebras, then for every $n \geq 0$, the diagram

$$\operatorname{HH}_n(\Gamma) \xrightarrow{j^{\otimes(n+1)}} \operatorname{HH}_n(\Lambda) \xrightarrow{\pi^{\otimes(n+1)}} \operatorname{HH}_n(\Gamma)$$

is a split pair of k-modules.

(ii) When the algebras involved have finite length over k, then so do the twisted bimodules induced by the automorphisms. Therefore the Hochschild homology groups also have finite length as k-modules.

Now consider a diagram of *two* split pairs

$$\begin{array}{ccccc} & & \Gamma & & \\ & & \downarrow{\scriptstyle j_\Gamma} & & \\ \Delta & \xrightarrow{j_\Delta} & \Lambda & \xrightarrow{\pi_\Delta} & \Delta \\ & & \downarrow{\scriptstyle \pi_\Gamma} & & \\ & & \Gamma & & \end{array}$$

of k-algebras running through Λ. Let $\Lambda \xrightarrow{\psi} \Lambda$ be an automorphism under which $\operatorname{Im} j_\Gamma$, $\operatorname{Im} j_\Delta$, $\operatorname{Ker}\pi_\Gamma$ and $\operatorname{Ker}\pi_\Delta$ are all invariant, and define the automorphisms $\Gamma \xrightarrow{\phi_\Gamma} \Gamma$ and $\Delta \xrightarrow{\phi_\Delta} \Delta$ by $\phi_\Gamma = \pi_\Gamma\psi j_\Gamma$ and $\phi_\Delta = \pi_\Delta\psi j_\Delta$, respectively. By

Proposition 3.1, for every $n \geq 0$ the above diagram induces a diagram

$$\begin{array}{ccccc} & & \mathrm{HH}_n(\Gamma, {}_{\phi_\Gamma}\Gamma_1) & & \\ & & \downarrow {\scriptstyle j_\Gamma^{\otimes(n+1)}} & & \\ \mathrm{HH}_n(\Delta, {}_{\phi_\Delta}\Delta_1) & \xrightarrow{j_\Delta^{\otimes(n+1)}} & \mathrm{HH}_n(\Lambda, {}_\psi \Lambda_1) & \xrightarrow{\pi_\Delta^{\otimes(n+1)}} & \mathrm{HH}_n(\Delta, {}_{\phi_\Delta}\Delta_1) \\ & & \downarrow {\scriptstyle \pi_\Gamma^{\otimes(n+1)}} & & \\ & & \mathrm{HH}_n(\Gamma, {}_{\phi_\Gamma}\Gamma_1) & & \end{array}$$

of split pairs. In particular, the k-modules $\mathrm{HH}_n(\Gamma, {}_{\phi_\Gamma}\Gamma_1)$ and $\mathrm{HH}_n(\Delta, {}_{\phi_\Delta}\Delta_1)$ are both direct summands of $\mathrm{HH}_n(\Lambda, {}_\psi \Lambda_1)$, thus it makes sense to compare them. The following result shows that, under certain conditions, these direct summands have trivial intersection. Consequently, we may identify the direct sum

$$\mathrm{HH}_n(\Gamma, {}_{\phi_\Gamma}\Gamma_1) \oplus \mathrm{HH}_n(\Delta, {}_{\phi_\Delta}\Delta_1)$$

with a submodule of $\mathrm{HH}_n(\Lambda, {}_\psi \Lambda_1)$. However, recall first the following notion. Let k be a commutative ring, and let Σ be a k-algebra given in terms of a ring homomorphism $k \xrightarrow{f} \Sigma$. Then Σ is an *augmented* k-algebra if there exists a homomorphism $\Sigma \xrightarrow{g} k$ of k-algebras such that the diagram

$$k \xrightarrow{f} \Sigma \xrightarrow{g} k$$

is a split pair of k-algebras (that is, the composition gf is the identity on k). We may then identify k with the subalgebra $\mathrm{Im}\, f$ of Σ, and $\Sigma = \mathrm{Im}\, f \oplus \mathrm{Ker}\, g$. The map g is the *augmentation map* of Σ, and the ideal $\mathrm{Ker}\, g$ is the *augmentation ideal*. The latter is denoted by $\mathfrak{a}_\Sigma$.

Theorem 3.3. *Let k be a commutative ring, and let $\Gamma \xrightarrow{j_\Gamma} \Lambda \xrightarrow{\pi_\Gamma} \Gamma$ and $\Delta \xrightarrow{j_\Delta} \Lambda \xrightarrow{\pi_\Delta} \Delta$ be two split pairs of augmented k-algebras, with the property that $j_\Gamma(\mathfrak{a}_\Gamma) \subseteq \mathrm{Ker}\, \pi_\Delta$. Furthermore, let $\Lambda \xrightarrow{\psi} \Lambda$ be an automorphism under which $\mathrm{Im}\, j_\Gamma, \mathrm{Im}\, j_\Delta, \mathrm{Ker}\, \pi_\Gamma$ and $\mathrm{Ker}\, \pi_\Delta$ are all invariant, and define $\Gamma \xrightarrow{\phi_\Gamma} \Gamma$ and $\Delta \xrightarrow{\phi_\Delta} \Delta$ by $\phi_\Gamma = \pi_\Gamma \psi j_\Gamma$ and $\phi_\Delta = \pi_\Delta \psi j_\Delta$, respectively. Then, viewed as submodules of $\mathrm{HH}_n(\Lambda, {}_\psi \Lambda_1)$, we have $\mathrm{HH}_n(\Gamma, {}_{\phi_\Gamma}\Gamma_1) \cap \mathrm{HH}_n(\Delta, {}_{\phi_\Delta}\Delta_1) = 0$, for $n \geq 1$.*

Proof. Let x be an element of $\mathrm{HH}_n(\Gamma, {}_{\phi_\Gamma}\Gamma_1)$, and write

$$x = \left(\sum_{i=1}^{t} \gamma^i \otimes \gamma_1^i \otimes \cdots \otimes \gamma_n^i \right) + \mathrm{Im}\, d_{n+1}$$

for some element $\sum_{i=1}^{t} \gamma^i \otimes \gamma_1^i \otimes \cdots \otimes \gamma_n^i \in \mathrm{Ker}\, d_n \subseteq {}_{\phi_\Gamma}\Gamma_1 \otimes \Gamma^{\otimes n}$. Then $j_\Gamma^{\otimes(n+1)}(x)$ is the element

$$\left(\sum_{i=1}^{t} j_\Gamma(\gamma^i) \otimes j_\Gamma(\gamma_1^i) \otimes \cdots \otimes j_\Gamma(\gamma_n^i) \right) + \mathrm{Im}\, d_{n+1}$$

of $\mathrm{HH}_n(\Lambda, {}_\psi \Lambda_1)$. Now every $\gamma \in \Gamma$ is of the form $\gamma = \alpha + a$ for some $\alpha \in k$ and $a \in \mathfrak{a}_\Gamma$. Then since $j_\Gamma(\mathfrak{a}_\Gamma)$ is contained in $\mathrm{Ker}\, \pi_\Delta$, we see that

$$j_\Gamma^{\otimes(n+1)}(x) = \alpha(1 \otimes \cdots \otimes 1) + w + \mathrm{Im}\, d_{n+1}$$

for some $\alpha \in k$ and $w \in \mathrm{Ker}\, \pi_\Delta^{\otimes(n+1)}$.

If n is odd, then $1 \otimes \cdots \otimes 1 = d_{n+1}(1 \otimes \cdots \otimes 1)$, and hence $j_\Gamma^{\otimes(n+1)}(x) = w + \mathrm{Im}\, d_{n+1}$. If n is even, then $d_n(1 \otimes \cdots \otimes 1) = 1 \otimes \cdots \otimes 1$, and so since $\alpha(1 \otimes \cdots \otimes 1) + w$ is an element of $\mathrm{Ker}\, d_n$, we see that $\alpha = 0$. Thus in this case we also obtain

$j_\Gamma^{\otimes(n+1)}(x) = w + \operatorname{Im} d_{n+1}$, hence $\pi_\Delta^{\otimes(n+1)} j_\Gamma^{\otimes(n+1)}(x) = 0$ for all $n \geq 1$. This shows that $\operatorname{HH}_n(\Gamma, {}_{\phi_\Gamma}\Gamma_1) \cap \operatorname{HH}_n(\Delta, {}_{\phi_\Delta}\Delta_1) = 0$ for $n \geq 1$. □

Corollary 3.4. *Let k be a commutative ring, and let $\Gamma \xrightarrow{j_\Gamma} \Lambda \xrightarrow{\pi_\Gamma} \Gamma$ and $\Delta \xrightarrow{j_\Delta} \Lambda \xrightarrow{\pi_\Delta} \Delta$ be two split pairs of augmented k-algebras, with the property that $j_\Gamma(\mathfrak{a}_\Gamma) \subseteq \operatorname{Ker} \pi_\Delta$. Then, viewed as submodules of $\operatorname{HH}_n(\Lambda)$, we have $\operatorname{HH}_n(\Gamma) \cap \operatorname{HH}_n(\Delta) = 0$, for $n \geq 1$.*

4. Applications

In this section we apply the results of the previous section, and obtain results and some interesting examples of the behavior of Hochschild homology.

We start with a result on different types of "products". Let k be a commutative ring, and Λ and Γ two augmented k-algebras with augmentation maps $\Lambda \xrightarrow{g_\Lambda} k$ and $\Gamma \xrightarrow{g_\Gamma} k$. Recall that the pullback of the diagram

$$\begin{array}{ccc} & & \Lambda \\ & & \downarrow{\scriptstyle g_\Lambda} \\ \Gamma & \xrightarrow{g_\Gamma} & k \end{array}$$

is called the *fibre product* of Λ and Γ (over k), and denoted by $\Lambda \times_k \Gamma$. Thus, the fibre product is isomorphic to the subalgebra $\{(\lambda, \gamma) \in \Lambda \oplus \Gamma \mid g_\Lambda(\lambda) = g_\Gamma(\gamma)\}$. Now let Σ be any k-algebra, and B be a Σ-Σ-bimodule. Recall that the k-module $\Sigma \oplus B$, endowed with the multiplication

$$(x_1, b_1)(x_2, b_2) \stackrel{\text{def}}{=} (x_1 x_2, x_1 b_2 + b_1 x_2),$$

is called the *trivial extension* of Σ with B, and denoted by $\Sigma \ltimes B$.

Theorem 4.1. *Let k be a commutative ring.*

1. *If Λ and Γ are augmented k-algebras, then $\operatorname{HH}_n(\Lambda) \oplus \operatorname{HH}_n(\Gamma)$ is a direct summand of $\operatorname{HH}_n(\Lambda \times_k \Gamma)$ for every $n \geq 1$.*
2. *If Λ is a k-algebra and B a bimodule, then $\operatorname{HH}_n(\Lambda)$ is a direct summand of $\operatorname{HH}_n(\Lambda \ltimes B)$ for every $n \geq 0$.*

Proof. For (1), we can write $\Lambda = k \oplus \mathfrak{a}_\Lambda$ and $\Gamma = k \oplus \mathfrak{a}_\Gamma$, where $\mathfrak{a}_\Lambda$ and $\mathfrak{a}_\Gamma$ are the augmentation ideals. Then the fibre product $\Lambda \times_k \Gamma$ is the algebra $k \oplus \mathfrak{a}_\Lambda \oplus \mathfrak{a}_\Gamma$, with $\mathfrak{a}_\Lambda \mathfrak{a}_\Gamma = 0 = \mathfrak{a}_\Gamma \mathfrak{a}_\Lambda$. The result now follows from Corollary 3.4. As for (2), note that there is an obvious split pair

$$\Lambda \to \Lambda \ltimes B \to \Lambda$$

of k-algebras, and so the result follows from Remarks 3.2. □

The next result is an application of split pairs to path algebras. Note that the algebras considered are not necessarily finitely generated over the commutative ground ring.

Theorem 4.2. *Let k be a commutative ring, let Q be an oriented quiver with a subquiver Q', and let I be an ideal in the path algebra kQ. Furthermore, let $J_{Q'}$ be the ideal in kQ/I generated by all the arrows not in Q'. Suppose that every path $p \in kQ'$ has the property that p is nonzero in $kQ'/(I \cap kQ')$ if and only if it is nonzero in $(kQ/I)/J_{Q'}$. Then $\operatorname{HH}_n(kQ'/(I \cap kQ'))$ is a direct summand of $\operatorname{HH}_n(kQ/I)$ for every $n \geq 0$.*

Proof. The assumptions given imply that there is a split pair

$$kQ'/(I \cap kQ') \to kQ/I \to kQ'/(I \cap kQ')$$

of k-algebras. The result now follows from Remarks 3.2. □

This theorem is particularly useful when studying the Hochschild homology of *monomial algebras*. Recall that these are algebras of the form kQ/I, where Q is an oriented quiver, and the defining ideal I is generated by (some) paths of length at least two. For such an algebra, given *any* subquiver Q' of Q, the hypothesis in Theorem 4.2 is satisfied. Consequently, for every $n \geq 0$, the homology group $\mathrm{HH}_n(kQ'/(I \cap kQ'))$ is a direct summand of $\mathrm{HH}_n(kQ/I)$. We record this in the following corollary.

Corollary 4.3. *Let k be a commutative ring, and kQ/I a monomial algebra with underlying quiver Q and defining ideal I. Then for any subquiver $Q' \subseteq Q$ and any $n \geq 0$, the homology group $\mathrm{HH}_n(kQ'/(I \cap kQ'))$ is a direct summand of $\mathrm{HH}_n(kQ/I)$.*

In the remaining part of this section, we shall need some well known results on the Hochschild homology of a truncated polynomial ring in one variable. For the convenience of the reader, we now recall these results. Fix a commutative ring k, and let Λ be a k-algebra which is projective as a k-module. Then for any bimodule B and every $n \geq 0$, the Hochschild homology group $\mathrm{HH}_n(\Lambda, B)$ is isomorphic to $\mathrm{Tor}_n^{\Lambda^{\mathrm{e}}}(B, \Lambda)$ as a k-module, where Λ^{e} is the enveloping algebra $\Lambda \otimes_k \Lambda^{\mathrm{op}}$ of Λ. Now consider the commutative k-algebra $A = k[x]/(x^a)$, where $a \geq 2$. This algebra is free as a k-module, hence the above applies. A minimal projective bimodule resolution of A is given by

$$\cdots \to A^{\mathrm{e}} \xrightarrow{\cdot(x\otimes 1 - 1 \otimes x)} A^{\mathrm{e}} \xrightarrow{\cdot\left(\sum_{i=0}^{a-1} x^{a-1-i} \otimes x^i\right)} A^{\mathrm{e}} \xrightarrow{\cdot(x\otimes 1 - 1 \otimes x)} A^{\mathrm{e}} \xrightarrow{\mu} A \to 0,$$

where μ is the multiplication map $u \otimes v \mapsto uv$. Given an A-A-bimodule B, the Hochschild homology group $\mathrm{HH}_n(A, B)$ is the nth homology group of the complex obtained by applying $B \otimes_{A^{\mathrm{e}}} -$ to the deleted projective resolution. Therefore, we see that $\mathrm{HH}_n(A, B)$ is the nth homology group of the two-periodic complex

$$\cdots \to B \xrightarrow{f} B \xrightarrow{g} B \xrightarrow{f} B \to 0,$$

in which the maps f and g are given by

$$\begin{aligned} f\colon b &\mapsto b \cdot x - x \cdot b \\ g\colon b &\mapsto \sum_{i=0}^{a-1} x^i \cdot b \cdot x^{a-1-i}. \end{aligned}$$

By taking $B = A$, we see immediately that $\mathrm{HH}_n(A) \neq 0$ for all $n \geq 0$. If moreover k is a field, then

$$\dim \mathrm{HH}_n(A) = \begin{cases} a & \text{when } n = 0 \\ a & \text{when } n > 0 \text{ and } \operatorname{char} k \text{ divides } a \\ a-1 & \text{when } n > 0 \text{ and } \operatorname{char} k \text{ does not divide } a. \end{cases}$$

Now suppose that k is a field and $\alpha \in k$ is a nonzero element which is *not* a root of unity, and denote by ψ the automorphism on A given by $x \mapsto \alpha x$. Then direct computation shows that, for $n \geq 1$, the Hochschild homology group $\mathrm{HH}_n(A, {}_{\psi}A_1)$ is given by

$$\dim \mathrm{HH}_n(A, {}_{\psi}A_1) = \begin{cases} 1 & \text{when } \alpha \text{ is a primitive } m\text{th rot of unity and } m|a \\ 0 & \text{otherwise.} \end{cases}$$

We summarize these facts on truncated polynomial rings in the following lemma.

Lemma 4.4. *Let k be a commutative ring, and A the algebra $k[x]/(x^a)$ for some $a \geq 2$. Then $\mathrm{HH}_n(A)$ is nonzero for every $n \geq 0$. If moreover k is a field, then the following hold:*

(1)

$$\dim \mathrm{HH}_n(A) = \begin{cases} a & \text{when } n = 0 \\ a & \text{when } n > 0 \text{ and } \operatorname{char} k \text{ divides } a \\ a-1 & \text{when } n > 0 \text{ and } \operatorname{char} k \text{ does not divide } a. \end{cases}$$

(2) *If $0 \neq \alpha \in k$ is not a root of unity, and $A \xrightarrow{\psi} A$ is the automorphism $x \mapsto \alpha x$, then*

$$\dim \mathrm{HH}_n(A, {}_{\psi}A_1) = \begin{cases} 1 & \text{when } \alpha \text{ is a primitive } m\text{th rot of unity and } m|a \\ 0 & \text{otherwise.} \end{cases}$$

We now use what we have just recalled above in the following result, which is no more than a special case of Theorem 4.2. It shows that the positive Hochschild homology groups of certain quotients of path algebras are always nonzero. To ease the notation, given an algebra Λ over some commutative ground ring, we define

$$\operatorname{HHdim} \Lambda \stackrel{\text{def}}{=} \sup\{n \in \mathbb{Z} \mid \mathrm{HH}_n(\Lambda) \neq 0\}.$$

This is the *Hochschild homology dimension* of Λ.

Theorem 4.5. *Let k be a commutative ring, let kQ be the path algebra of an oriented quiver Q, and let $I \subseteq kQ$ be an ideal. Suppose Q contains a loop y with the property that, for every $i \geq 1$, the element y^i is nonzero in kQ/I if and only if it is nonzero in $(kQ/I)/J_y$, where J_y is the ideal in kQ/I generated by all the arrows except y. Moreover, suppose y is nilpotent with nilpotency index $a > 1$. Then $\operatorname{HHdim} KQ/I = \infty$. In particular, if k is a field, then*

$$\dim \mathrm{HH}_n(kQ/I) \geq \begin{cases} a & \text{when } \operatorname{char} k \text{ divides } a \\ a-1 & \text{otherwise} \end{cases}$$

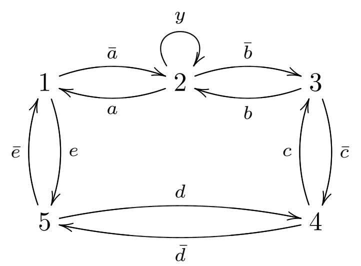
for all $n \geq 1$.

Proof. The given assumptions imply that there is a split pair

$$k[y]/(y^a) \to kQ/I \to k[y]/(y^a)$$

of k-algebras, and so the result follows from Remarks 3.2 and Lemma 4.4. ☐

Example. Let k be a commutative ring and Q the quiver



Furthermore, let $I \subseteq kQ$ be the ideal generated by the elements

$$y^t, \quad \bar{a}a + b\bar{b}, \quad \bar{b}b + c\bar{c}, \quad \bar{c}c + d\bar{d}, \quad \bar{d}d + e\bar{e}, \quad \bar{e}e + a\bar{a}$$

for some $t \geq 2$. Then $\mathrm{HH}_n(kQ/I)$ is nonzero for all n. In particular, if k is a field, then

$$\dim \mathrm{HH}_n(kQ/I) \geq \begin{cases} t & \text{when } n = 0 \\ t & \text{when } n > 0 \text{ and } \operatorname{char} k \text{ divides } a \\ t-1 & \text{otherwise.} \end{cases}$$

A special case occurs when the underlying quiver contains a loop which squares to zero in the path algebra modulo relations.

Corollary 4.6. *Let k be a commutative ring, let kQ be the path algebra of an oriented quiver Q, and let $I \subseteq kQ$ be an ideal. If Q contains a loop $y \notin I$ with the property that $y^2 = 0$ in kQ/I, then* $\operatorname{HHdim} KQ/I = \infty$.

We next apply our results to graded algebras. Let k be a field and $A = \oplus_{i=0}^{\infty} A_i$ a graded k-algebra. Then A is *connected* if $A_0 = k$, and *graded-commutative* if $xy = (-1)^{|x||y|}yx$ for all homogeneous elements $x, y \in A$, where $|x|$ and $|y|$ denote the degrees of these elements. The following result shows that if a graded-commutative connected algebra has nonzero elements of odd degree, then its Hochschild dimension is infinite.

Theorem 4.7. *Let k be a field and $A = \oplus_{i=0}^{\infty} A_i$ a graded-commutative connected k-algebra. If either*

(1) $\operatorname{char} k \neq 2$ *and there exists an odd integer j with $A_j \neq 0$, or*
(2) *A is finitely generated and either* $\operatorname{char} k = 2$ *or* $A = \oplus_{i=0}^{\infty} A_{2i}$,

then $\operatorname{HHdim} A = \infty$.

Proof. If (1) holds, then let j be the smallest odd integer with $A_j \neq 0$, and let $x \in A_j$ be part of a k-basis for A_j. Then $x^2 = 0$, and there is a split pair

$$k[x]/(x^2) \to A \to k[x]/(x^2)$$

of k-algebras. The rightmost map is obtained by taking the quotient of A with all the A_i for $i \neq j$, and all the elements in a k-basis for A_j different from x. Thus it follows from Remarks 3.2 and Lemma 4.4 that $\operatorname{HHdim} A = \infty$ in this case. If (2) holds, then our algebra is commutative and finitely generated, hence $\operatorname{HHdim} A = \infty$ by [AV-P]. □

Corollary 4.8. *If k is a field and A a graded-commutative connected Noetherian k-algebra, then* $\operatorname{HHdim} A = \infty$.

We now turn our attention to a class of algebras which are noncommutative analogues of truncated polynomial rings. Let k be a field, let $c \geq 1$ be an integer, and let $\mathbf{q} = (q_{ij})$ be a $c \times c$ commutation matrix with entries in k. That is, the diagonal entries q_{ii} are all 1, and $q_{ij}q_{ji} = 1$ for $i \neq j$. Furthermore, let $\mathbf{a}_c = (a_1, \dots, a_c)$ be an ordered sequence of c integers with $a_i \geq 2$. The *quantum complete intersection* $A_{\mathbf{q}}^{\mathbf{a}_c}$ determined by these data is the algebra

$$A_{\mathbf{q}}^{\mathbf{a}_c} \stackrel{\text{def}}{=} k\langle x_1, \dots, x_c \rangle / (x_i^{a_i}, x_i x_j - q_{ij} x_j x_i),$$

which is finite dimensional of dimension $\prod a_i$. The image of x_i in this quotient will also be denoted by x_i. The following result gives a lower bound for the dimension of the Hochschild homology groups of this algebra.

Proposition 4.9. *For each $n \geq 1$, the inequality*

$$\dim \operatorname{HH}_n(A_{\mathbf{q}}^{\mathbf{a}_c}) \geq \sum_{i=1}^{c} a_i - c + \#\{a_i \mid \operatorname{char} k \text{ divides } a_i\}$$

holds.

Proof. For each $1 \leq u \leq c$, let A_u be the subalgebra of $A_{\mathbf{q}}^{\mathbf{a}_c}$ generated by x_u. Then $A_u = k[x_u]/(x_u^{a_u})$, and there is a split pair

$$A_u \xrightarrow{j_u} A_{\mathbf{q}}^{\mathbf{a}_c} \xrightarrow{\pi_u} A_u$$

of k-algebras. The augmentation ideal in A_u is just the radical (x_u), and $j_u(x_u)$ is contained in $\operatorname{Ker} \pi_v$ whenever $u \neq v$. Therefore, from Corollary 3.4, we see that $\operatorname{HH}_n(A_u) \cap \operatorname{HH}_n(A_v) = 0$ for $u \neq v$, when wiewing $\operatorname{HH}_n(A_u)$ and $\operatorname{HH}_n(A_v)$ as subspaces of $\operatorname{HH}_n(A_{\mathbf{q}}^{\mathbf{a}_c})$. Hence $\operatorname{HH}_n(A_1) \oplus \cdots \oplus \operatorname{HH}_n(A_c)$ is a subspace of $\operatorname{HH}_n(A_{\mathbf{q}}^{\mathbf{a}_c})$, and the result follows from Lemma 4.4. □

As for the Hochschild cohomology groups of the quantum complete intersections, these can be computed using Hochschild homology. Recall that if k is a field and Λ a finite dimensional k-algebra with a bimodule B, then the nth *Hochschild cohomology* group of Λ, with coefficients in B, is the vector space

$$\operatorname{HH}^n(\Lambda, B) \stackrel{\text{def}}{=} \operatorname{Ext}^n_{\Lambda^e}(\Lambda, B).$$

When $B = \Lambda$, we write $\operatorname{HH}^n(\Lambda)$ for $\operatorname{HH}^n(\Lambda, \Lambda)$. From [CaE, VI.5.3] it follows that $D(\operatorname{HH}^n(\Lambda, B)) \simeq \operatorname{HH}_n(\Lambda, D(B))$, where D denotes the k-dual $\operatorname{Hom}_k(-, k)$. In particular, the dimension of $\operatorname{HH}^n(\Lambda, B)$ equals that of $\operatorname{HH}_n(\Lambda, D(B))$. Now, a quantum complete intersection $A^{\mathbf{a}_c}_{\mathbf{q}}$ is Frobenius, with an isomorphism $A^{\mathbf{a}_c}_{\mathbf{q}} \xrightarrow{\phi} D(A^{\mathbf{a}_c}_{\mathbf{q}})$ of left modules given by

$$\phi(1) \colon \sum_{i_1, \dots, i_c} \alpha_{i_1, \dots, i_c} x_c^{i_c} \cdots x_1^{i_1} \mapsto \alpha_{a_1 - 1, \dots, a_c - 1}.$$

That is, the element $\phi(1)$ maps an element $w \in A^{\mathbf{a}_c}_{\mathbf{q}}$ to the coefficient of the socle element $x_c^{a_c - 1} \cdots x_1^{a_1 - 1}$ in w. The Nakayama automorphism $A^{\mathbf{a}_c}_{\mathbf{q}} \xrightarrow{\nu} A^{\mathbf{a}_c}_{\mathbf{q}}$, with the defining property that $w \cdot \phi(1) = \phi(1) \cdot \nu(w)$ for all $w \in A^{\mathbf{a}_c}_{\mathbf{q}}$, is then given by

$$x_u \mapsto \prod_{i=1}^{c} q_{iu}^{a_i - 1} x_u$$

(cf. [Ber, Lemma 3.1]). The bimodules $D(A^{\mathbf{a}_c}_{\mathbf{q}})$ and ${}_{\nu}(A^{\mathbf{a}_c}_{\mathbf{q}})_1$ are isomorphic, hence

$$\dim \operatorname{HH}^n(A^{\mathbf{a}_c}_{\mathbf{q}}) = \dim \operatorname{HH}_n\left(A^{\mathbf{a}_c}_{\mathbf{q}}, {}_{\nu}(A^{\mathbf{a}_c}_{\mathbf{q}})_1\right)$$

from the above arguments. Using Theorem 3.3, we then obtain the following result.

Proposition 4.10. *For each $1 \le u \le c$, denote the element $\prod_{i=1}^{c} q_{iu}^{a_i - 1}$ by α_u, and let A_u be the subalgebra of $A^{\mathbf{a}_c}_{\mathbf{q}}$ generated by x_u. Then*

$$\dim \operatorname{HH}^n(A^{\mathbf{a}_c}_{\mathbf{q}}) \ge \sum_{u=1}^{c} \dim \operatorname{HH}_n(A_u, {}_{\phi_u}(A_u)_1),$$

where $A_u \xrightarrow{\phi_u} A_u$ is the automorphism given by $x_u \mapsto \alpha_u x_u$.

The Hochschild homology and cohomology groups of an arbitrary quantum complete intersection have been completely described in [Opp]. As we saw in Proposition 4.9, the homology groups never vanish. However, the vanishing of the cohomology groups $\operatorname{HH}^n(A^{\mathbf{a}_c}_{\mathbf{q}})$ depend heavily on the commutators q_{ij}, as the following examples illustrate.

Examples. (i) Let A be the quantum complete intersection

$$k\langle x, y\rangle / (x^a, xy - qyx, y^b),$$

where $a, b \ge 2$. If q is not a root of unity, then $\operatorname{HH}^n(A) = 0$ for $n \ge 3$, as was shown in [BeE] (the special case when $a = b = 2$ was treated in [BGMS]).

(ii) Suppose all the commutators q_{ij} are roots of unity. Then $\operatorname{HH}^n(A^{\mathbf{a}_c}_{\mathbf{q}})$ is nonzero for all n, as was shown in [BeO].

(iii) Let A be the quantum complete intersection with $c = 4$ and commutator matrix given by

$$\mathbf{q} = \begin{pmatrix} 1 & q_{21} & 1 & 1 \\ q_{12} & 1 & 1 & 1 \\ 1 & 1 & 1 & q_{34} \\ 1 & 1 & q_{43} & 1 \end{pmatrix}$$

Then $A = A_{12} \otimes_k A_{34}$, where A_{ij} is the quantum complete intersection

$$A_{ij} = k\langle x_i, x_j\rangle / (x_i^{a_i}, x_i x_j - q_{ij} x_j x_i, x_j^{a_j}).$$

The Hochschild cohomology of A decomposes as

$$\mathrm{HH}^n(A) = \oplus_{i=0}^n \mathrm{HH}^i(A_{12}) \otimes_k \mathrm{HH}^{n-i}(A_{34}),$$

and so from the above examples we see that $\mathrm{HH}^n(A) \neq 0$ for $n \geq 5$ if and only if one of q_{12}, q_{34} is a root of unity.

(iv) Suppose a column (or, equivalently, a row), say column u, in the commutation matrix $\mathbf{q}$ consists entirely of 1s, that is, $q_{iu} = 1$ for $1 \leq i \leq c$. Then $\nu(x_u) = x_u$, and so from Proposition 4.10 we see that

$$\dim \mathrm{HH}^n(A_{\mathbf{q}}^{\mathbf{a}_c}) \geq \begin{cases} a_u & \text{when } \operatorname{char} k \text{ divides } a_u \\ a_u - 1 & \text{otherwise} \end{cases}$$

for all $n \geq 1$.

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