

HOCHSCHILD HOMOLOGY AND TRUNCATED CYCLES

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Dedicated to Professor Claus Michael Ringel on the occasion of his 65th birthday

ABSTRACT. We study algebras having 2-truncated cycles, and show that these algebras have infinitely many nonzero Hochschild homology groups. Consequently, algebras of finite global dimension have no 2-truncated cycles, and therefore satisfy a higher version of the “no loops conjecture”.

1. INTRODUCTION

In his 1989 paper [Hap], Happel noted that if a finite dimensional algebra over an algebraically closed field is of finite global dimension, then all its higher Hochschild cohomology groups vanish. He then remarked that the converse was unknown, and this gave birth to “Happel’s question”. Namely, if all the higher Hochschild cohomology groups vanish for a finite dimensional algebra (not necessarily over an algebraically closed ground field), then must the algebra have finite global dimension? A positive answer for commutative algebras was provided in [AvI]. However, it was shown in [BGMS] that this does not hold in general.

The homology version of Happel’s question, first conjectured by the second author in [Han], is open: if all the higher Hochschild homology groups of a finite dimensional algebra vanish, then is the algebra of finite global dimension? As in the cohomology case, the answer is yes for commutative algebras (cf. [AV-P]). Moreover, the conjecture has been established for large classes of noncommutative algebras in [BM1], [BM2], [Han] and [SV-P], for example monomial algebras, Koszul algebras, cellular algebras and local graded algebras.

In this paper, we focus on the combinatorics of quivers of algebras. We show that the conjecture holds for all algebras having 2-truncated cycles, that is, cycles in which the product of two consecutive arrows is always zero. In other words, we show that such algebras have infinitely many nonzero Hochschild homology groups. Consequently, a higher version of the well-known “no loops conjecture” holds: algebras of finite global dimension have no 2-truncated cycles.

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2. HOCHSCHILD HOMOLOGY

Let k be a commutative ring and A a k -algebra. Thus, there exists a nonzero ring homomorphism $k \rightarrow A$, whose image is contained in the center of A . We do not assume that A is finitely generated as a k -module.

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For each $n \geq 1$, denote by $A^{\otimes n}$ the n -fold tensor product $A \otimes_k \cdots \otimes_k A$, in which there are n copies of A . Moreover, consider the map $A^{\otimes(n+1)} \xrightarrow{d_n} A^{\otimes n}$ given by

$$a_0 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

Then $d_n d_{n+1} = 0$, and so we may form a complex

$$\cdots \rightarrow A^{\otimes 4} \xrightarrow{d_3} A^{\otimes 3} \xrightarrow{d_2} A^{\otimes 2} \xrightarrow{d_1} A,$$

in which $A^{\otimes n}$ is in degree $n-1$. The n th homology group of this complex, denoted $\mathrm{HH}_n(A)$, is the n th *Hochschild homology* group of our algebra A . Thus $\mathrm{HH}_n(A)$ is the abelian group

$$\mathrm{Ker} \left(A^{\otimes(n+1)} \xrightarrow{d_n} A^{\otimes n} \right) / \mathrm{Im} \left(A^{\otimes(n+2)} \xrightarrow{d_{n+1}} A^{\otimes(n+1)} \right),$$

where d_0 is the zero map. This is a k -module, in general not finitely generated since A is not assumed to be module finite over k . We refer to the above complex as the *Hochschild complex* of A . The main emphasis of this paper is on algebras having infinitely many nonzero Hochschild homology groups. We therefore define

$$\mathrm{HHdim} A \stackrel{\mathrm{def}}{=} \sup \{ n \in \mathbb{Z} \mid \mathrm{HH}_n(A) \neq 0 \}.$$

This is the *Hochschild homology dimension* of the algebra A .

Hochschild homology (unlike Hochschild cohomology) is functorial in the category of k -algebras. Namely, let A and B be two k -algebras, and suppose that $A \xrightarrow{f} B$ is a k -algebra homomorphism. Then f is a homomorphism of rings, and there is a commutative diagram

$$\begin{array}{ccc} & k & \\ & \swarrow & \searrow \\ A & \xrightarrow{f} & B \end{array}$$

of maps, where $k \rightarrow A$ and $k \rightarrow B$ are the maps giving the k -algebra structure of A and B . This implies that for each $n \geq 1$, the diagram

$$\begin{array}{ccc} A^{\otimes(n+1)} & \xrightarrow{d_n} & A^{\otimes n} \\ \downarrow f^{\otimes(n+1)} & & \downarrow f^{\otimes n} \\ B^{\otimes(n+1)} & \xrightarrow{d_n} & B^{\otimes n} \end{array}$$

commutes, giving a map

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^{\otimes 4} & \xrightarrow{d_3} & A^{\otimes 3} & \xrightarrow{d_2} & A^{\otimes 2} & \xrightarrow{d_1} & A \\ & & \downarrow f^{\otimes 4} & & \downarrow f^{\otimes 3} & & \downarrow f^{\otimes 2} & & \downarrow f \\ \cdots & \longrightarrow & B^{\otimes 4} & \xrightarrow{d_3} & B^{\otimes 3} & \xrightarrow{d_2} & B^{\otimes 2} & \xrightarrow{d_1} & B \end{array}$$

of Hochschild complexes. For each $n \geq 0$, this map of Hochschild complexes induces a map $\mathrm{HH}_n(A) \xrightarrow{f^{\otimes(n+1)}} \mathrm{HH}_n(B)$ of Hochschild homology groups, given by

$$\sum_i a_0^i \otimes \cdots \otimes a_n^i + \mathrm{Im} d_{n+1} \mapsto \sum_i f(a_0^i) \otimes \cdots \otimes f(a_n^i) + \mathrm{Im} d_{n+1}.$$

This simple fact is the key ingredient in the proof of the main theorem, where we want to show that a certain Hochschild homology group, say $\mathrm{HH}_n(A)$, is nonzero. Namely, we construct an algebra homomorphism $A \rightarrow B$, where B is an algebra

which has Hochschild homology groups that are more amenable to computation. If we can show that the image of $\mathrm{HH}_n(A) \rightarrow \mathrm{HH}_n(B)$ is nonzero, then this forces $\mathrm{HH}_n(A)$ to be nonzero.

3. TRUNCATED CYCLES

In this final section we prove the main result, which shows that certain algebras have nonzero Hochschild homology groups in arbitrarily high degrees. We formulate this result for path algebras of quivers. Recall that if k is a commutative ring and Q a quiver with a finite number of vertices $\{1, \dots, n\}$, then the path algebra kQ is the unital k -algebra defined as follows. As a k -module, the algebra kQ is free with basis e_1, \dots, e_n and all possible finite sequences

$$i_1 \xrightarrow{x_1} i_2 \xrightarrow{x_2} i_3 \rightarrow \dots \rightarrow i_w \xrightarrow{x_w} i_{w+1}$$

of arrows. We write the latter as $x_w \cdots x_1$, denote its *initial vertex* i_1 by $s(x_w \cdots x_1)$, and its *terminal vertex* i_{w+1} by $t(x_w \cdots x_1)$. Moreover, we view each e_i as a trivial path, with $s(e_i) = i = t(e_i)$. If p is any path and $1 \leq i \leq n$, then multiplication of p and e_i is defined as

$$p \cdot e_i = \begin{cases} p & \text{if } s(p) = i \\ 0 & \text{if } s(p) \neq i, \end{cases} \quad e_i \cdot p = \begin{cases} p & \text{if } t(p) = i \\ 0 & \text{if } t(p) \neq i. \end{cases}$$

Finally, if p and q are nontrivial paths, then their product is defined as

$$p \cdot q = \begin{cases} pq & \text{if } s(p) = t(q) \\ 0 & \text{if } s(p) \neq t(q), \end{cases}$$

i.e. the concatenation of the two paths when possible. Thus the unit in kQ is $e_1 + \dots + e_n$, and the elements e_1, \dots, e_n are orthogonal idempotents. We denote the two-sided ideal generated by the arrows by J_{kQ} .

The importance of path algebras is established in a classical result of Gabriel. Recall that an ideal I in a path algebra kQ is *admissible* if $J_{kQ}^n \subseteq I \subseteq J_{kQ}^2$ for some $n \geq 2$. Gabriel showed that every finite dimensional basic algebra over an algebraically closed field k is isomorphic to a factor kQ/I of a path algebra, where Q is some finite quiver and $I \subseteq kQ$ is an admissible ideal (cf. [ASS, Theorem II.3.7]). Moreover, every finite dimensional algebra over a field is Morita equivalent to a finite dimensional basic algebra over the same field.

If $I \subseteq J_{kQ}^2$ is an ideal in the path algebra kQ , then a *cycle* in kQ/I is a finite sequence x_1, \dots, x_u of arrows with the property that $t(x_i) = s(x_{i+1})$ and $t(x_u) = s(x_1)$. Every algebra kQ/I of infinite global dimension over a field contains a cycle, this follows from [ENN, Theorem 8].

Definition. A cycle x_1, \dots, x_u in kQ/I is *m -truncated* (for an integer $m \geq 2$) if

$$x_{i+m-1} \cdots x_{i+1} x_i = 0 \quad \text{and} \quad x_{i+m-2} \cdots x_{i+1} x_i \neq 0$$

for all i (where the indices are modulo u).

The main result deals with path algebras whose quivers contain 2-truncated cycles, that is, cycles in which the product of any two consecutive arrows is zero. Note that the algebras considered are *not* necessarily finitely generated over the commutative ground ring: although we assume the quiver has finitely many vertices, there is no assumption on the number of arrows. In the special case when the algebra is local, the theorem provides a generalization of [SV-P, Theorem II].

Theorem 3.1. *Let k be a commutative ring, Q a quiver with a finite number of vertices, and $I \subseteq kQ$ an ideal contained in J_{kQ}^2 . Suppose kQ/I contains a 2-truncated cycle x_1, \dots, x_u . Then for every $n \geq 1$ with $un \equiv u \pmod{2}$, the element*

$$(x_u \otimes \cdots \otimes x_1)^{\otimes n}$$

represents a nonzero element in $\mathrm{HH}_{un-1}(kQ/I)$. In particular, $\mathrm{HHdim} kQ/I = \infty$.

Proof. We denote the algebra kQ/I by A . We may assume that the 2-truncated cycle is minimal, i.e. that $x_i \neq x_j$ whenever $i \neq j$: if not, there is a smaller 2-truncated cycle.

Consider the Hochschild homology complex

$$\dots \rightarrow A^{\otimes(un+1)} \xrightarrow{d_{un}} A^{\otimes un} \xrightarrow{d_{un-1}} A^{\otimes(un-1)} \rightarrow \dots$$

and the element $(x_u \otimes \dots \otimes x_1)^{\otimes n}$ in $A^{\otimes un}$. Since the cycle is 2-truncated, this element belongs to $\mathrm{Ker} d_{un-1}$, hence it represents an element $\xi \in \mathrm{HH}_{un-1}(A)$. Let B be the algebra $B = kQ/J_{kQ}^2$, and consider the natural surjective k -algebra homomorphism $A \xrightarrow{f} B$. The image of ξ under the map $\mathrm{HH}_{un-1}(A) \xrightarrow{f^{\otimes un}} \mathrm{HH}_{un-1}(B)$ is represented by $(x_u \otimes \dots \otimes x_1)^{\otimes n} \in B^{\otimes un}$. If $(x_u \otimes \dots \otimes x_1)^{\otimes n} \in B^{\otimes un}$ represents a non-zero element in $\mathrm{HH}_{un-1}(B)$, then $(x_u \otimes \dots \otimes x_1)^{\otimes n} \in A^{\otimes un}$ represents a non-zero element in $\mathrm{HH}_{un-1}(A)$.

The algebra B has a grading $B = B_0 \oplus B_1$ with all arrows having degree one. The degree un layer of the Hochschild complex has the following shape.

$$\dots \rightarrow (B^{\otimes un+1})_{un} \xrightarrow{d_{un}} (B^{\otimes un})_{un} \xrightarrow{d_{un-1}} 0 \rightarrow \dots$$

The k -module $(B^{\otimes un})_{un}$ is generated by elements of the form

$$z_1 \otimes \dots \otimes z_{un},$$

where each z_j is an arrow. As a k -module $(B^{\otimes un})_{un}$ decomposes as

$$(B^{\otimes un})_{un} = W \oplus U,$$

where W is the submodule of $(B^{\otimes un})_{un}$ generated by those generators $z_1 \otimes \dots \otimes z_{un}$ having the property that the path $z_1 \dots z_{un}$ is a cyclic shift of the path $(x_u \dots x_1)^n$ in kQ , and U is the submodule generated by all remaining generators. Similarly, the k -module $(B^{\otimes un+1})_{un}$ is generated by elements of the form $z_0 \otimes \dots \otimes z_{un}$, where exactly one z_j is a trivial path and all other z_j are arrows. As a k -module $(B^{\otimes un+1})_{un}$ decomposes as

$$(B^{\otimes un+1})_{un} = W' \oplus U',$$

where W' is the submodule of $(B^{\otimes un+1})_{un}$ generated by those generators $z_0 \otimes \dots \otimes z_{un}$ having the property that the path $z_0 \dots z_{un}$ is a cyclic shift of the path $(x_u \dots x_1)^n$ in kQ , and U' is the submodule generated by all remaining generators. We have $d_{un}(W') \subseteq W$ and $d_{un}(U') \subseteq U$.

We must show that $(x_u \otimes \dots \otimes x_1)^{\otimes n} \in B^{\otimes un}$ does not belong to $\mathrm{Im} d_{un}$. Suppose therefore there exists an element $w \in B^{\otimes(un+1)}$ with the property that $(x_u \otimes \dots \otimes x_1)^{\otimes n} = d_{un}(w)$. Since

$$(x_u \otimes \dots \otimes x_1)^{\otimes n} \in W \subseteq (B^{\otimes un})_{un},$$

we may assume

$$w \in W' \subseteq (B^{\otimes un+1})_{un}.$$

Thus we can write w as a k -linear combination of elements of the form $z_0 \otimes \dots \otimes z_{un}$, in which exactly one z_j is a trivial path and all other z_j are arrows, and $z_0 \dots z_{un}$ is a cyclic shift of $(x_u \dots x_1)^n$. In each term we may assume z_0 is the trivial path, since $d_{un}(z_0 \otimes \dots \otimes z_{un}) = 0$ in all other cases.

From what we have shown so far, we may assume w can be written as

$$\begin{aligned} w &= b_1 \otimes (x_u \otimes \cdots \otimes x_1)^{\otimes n} + \\ &\quad b_2 \otimes (x_1 \otimes x_u \otimes \cdots \otimes x_2)^{\otimes n} + \\ &\quad \vdots \\ &\quad b_u \otimes (x_{u-1} \otimes \cdots \otimes x_1 \otimes x_u)^{\otimes n}, \end{aligned}$$

where for $1 \leq i \leq u$ we have $b_i = \alpha_i e_i$ for some $\alpha_i \in k$. When we apply d_{un} to $b_1 \otimes (x_u \otimes \cdots \otimes x_1)^{\otimes n}$, the result is just

$$\alpha_1 [(x_u \otimes \cdots \otimes x_1)^{\otimes n} + (-1)^{un} (x_1 \otimes x_u \otimes \cdots \otimes x_2)^{\otimes n}],$$

and similarly for the other tensor elements in the expression for w . This shows

$$\begin{aligned} d_{un}(w) &= \alpha_1 [(x_u \otimes \cdots \otimes x_1)^{\otimes n} + (-1)^{un} (x_1 \otimes x_u \otimes \cdots \otimes x_2)^{\otimes n}] + \\ &\quad \alpha_2 [(x_1 \otimes x_u \otimes \cdots \otimes x_2)^{\otimes n} + (-1)^{un} (x_2 \otimes x_1 \otimes x_u \otimes \cdots \otimes x_3)^{\otimes n}] + \\ &\quad \vdots \\ &\quad \alpha_u [(x_{u-1} \otimes \cdots \otimes x_1 \otimes x_u)^{\otimes n} + (-1)^{un} (x_u \otimes \cdots \otimes x_1)^{\otimes n}], \end{aligned}$$

where each α_i is an element of k . The elements α_i must satisfy equations

$$\begin{aligned} \alpha_1 + (-1)^{un} \alpha_u &= 1 \\ \alpha_2 + (-1)^{un} \alpha_1 &= 0 \\ \alpha_3 + (-1)^{un} \alpha_2 &= 0 \\ &\vdots \\ \alpha_u + (-1)^{un} \alpha_{u-1} &= 0, \end{aligned}$$

but this is impossible when n satisfies $un \equiv u \pmod{2}$. Consequently, for such n , the element $(x_u \otimes \cdots \otimes x_1)^{\otimes n} \in B^{\otimes un}$ represents a nonzero element in $\mathrm{HH}_{un-1}(B)$. It follows that $(x_u \otimes \cdots \otimes x_1)^{\otimes n} \in A^{\otimes un}$ represents a nonzero element in $\mathrm{HH}_{un-1}(A)$, and this finishes the proof. \square

From now on we focus on finite dimensional algebras. As mentioned prior to Theorem 3.1, when the ground field is algebraically closed, then every such algebra is Morita equivalent to a factor of a path algebra. We start with algebras whose radicals square to zero. The Hochschild homology groups of such algebras arising from quivers were computed in [Skö] and [Cib].

Corollary 3.2. *Let A be a finite dimensional algebra over an algebraically closed field. If A has infinite global dimension and its radical squares to zero, then $\mathrm{HHdim} A = \infty$.*

Proof. We may assume that A is a factor of a path algebra of a finite quiver. Since its global dimension is infinite, the quiver has a cycle, which must be 2-truncated since the radical of the algebra squares to zero. \square

For finite dimensional algebras, or more generally, for algebras which are projective over the commutative ground ring, the Hochschild homology groups are isomorphic to the bimodule homology groups (cf. [CaE, p. 175]). In other words, if k is a field and A is a finite dimensional k -algebra, then $\mathrm{HH}_n(A) \simeq \mathrm{Tor}_n^{A \otimes_k A^{\mathrm{op}}}(A, A)$ for every n . If moreover A is separable over k , then there is an equality $\mathrm{gldim} A = \mathrm{pd}_{A \otimes_k A^{\mathrm{op}}} A$ (cf. [Hap, p. 110]), and so in this case $\mathrm{HH}_n(A)$ vanishes for large n whenever the global dimension of A is finite. Now let Q be a finite quiver and I an admissible ideal in the path algebra kQ . The finite dimensional path algebra kQ/I

is then separable over k , for any field k . We therefore obtain the following corollary to Theorem 3.1.

Corollary 3.3. *Let k be a field, Q a finite quiver and I an admissible ideal in kQ . If the algebra kQ/I has finite global dimension, then it contains no 2-truncated cycle.*

This can be thought of as a higher version of the well-known “no loops conjecture”, which says that kQ/I has no loop whenever the global dimension is finite. This conjecture was proved in [Igu] for a large class of finite dimensional algebras, including all those for which the ground field is algebraically closed.

It is natural to ask whether Corollary 3.3 still holds if “2-truncated” is replaced by “ m -truncated”: if the algebra kQ/I has finite global dimension, can it contain an m -truncated cycle for some $m \geq 2$? We end this paper by establishing this for monomial algebras.

Proposition 3.4. *Let k be a field, Q a finite quiver and I an admissible ideal in kQ such that kQ/I is a monomial algebra. If the global dimension of kQ/I is finite, then it contains no m -truncated cycle for any $m \geq 2$.*

Proof. Suppose the algebra does contain an m -truncated cycle. Then there exists a truncated basic cycle algebra overlying kQ/I , and this is a selfinjective Nakayama algebra of infinite global dimension. By [IgZ, Proposition 3.2], the algebra kQ/I is also of infinite global dimension. \square

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