Support and rank varieties for quantum complete intersections

Petter Andreas Bergh
(joint work with Karin Erdmann)

Support varieties for modules over finite dimensional algebras were introduced in [6], using Hochschild cohomology. As shown in [5], when certain finiteness conditions hold, the theory is very similar to the theory of cohomological support varieties for modules over group algebras and commutative complete intersections.

Fix a field k. Let Λ be a finite dimensional k-algebra with radical $\mathfrak{r}$. The Hochschild cohomology ring $\operatorname{HH}^*(\Lambda)$ is graded commutative, and for every left Λ-module M there is a homomorphism

$$\operatorname{HH}^*(\Lambda) \xrightarrow{\varphi_M = - \otimes_\Lambda M} \operatorname{Ext}^*_\Lambda(M,M)$$

of graded rings. The Hochschild cohomology ring acts graded-commutatively on cohomology groups; for any Λ-module N and homogeneous elements $\eta \in \operatorname{HH}^*(\Lambda)$ and $\theta \in \operatorname{Ext}^*_\Lambda(M,N)$, the equality

$$\varphi_N(\eta) \cdot \theta = (-1)^{|\eta||\theta|} \theta \cdot \varphi_M(\eta)$$

holds.

Definition. Given a commutative graded subalgebra $H \subseteq \operatorname{HH}^*(\Lambda)$, the *support variety* of an ordered pair (M,N) of Λ-modules, with respect to H, is

$$\operatorname{V}_H(M,N) \overset{\text{def}}{=} \{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \operatorname{Ann}_H\left(\operatorname{Ext}^*_\Lambda(M,N)\right) \subseteq \mathfrak{m}\}.$$

The support variety $\operatorname{V}_H(M)$ of a module is defined to be $\operatorname{V}_H(M,M)$; it is not difficult to show that $\operatorname{V}_H(M)$ equals both $\operatorname{V}_H(M,\Lambda/\mathfrak{r})$ and $\operatorname{V}_H(\Lambda/\mathfrak{r},M)$. The following theorem summarizes the most important properties. Recall that the complexity $\operatorname{cx} M$ of M is the rate of growth of its minimal projective resolution, whereas the plexity $\operatorname{px} M$ is the rate of growth of its minimal injective resolution.

Theorem 1 ([5]). *Suppose H is Noetherian and $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ is a finitely generated H-module.*

(1) *For all Λ-modules M,N the H-module $\operatorname{Ext}^*_\Lambda(M,N)$ is finitely generated.*
(2) *Λ is Gorenstein.*
(3) *The equalities $\dim \operatorname{V}_H(M) = \operatorname{cx} M = \operatorname{px} M$ hold. In particular, a module has finite projective (injective) dimension if and only if its support variety is trivial.*
(4) *$\dim \operatorname{V}_H(M) = 1$ if and only if M is eventually periodic.*
(5) *If V is a homogeneous subvariety of $\operatorname{MaxSpec} H$, then there exists a Λ-module M with $\operatorname{V}_H(M) = V$.*
(6) *Suppose Λ is selfinjective and $\operatorname{V}_H(M) = V_1 \cup V_2$ with V_1, V_2 homogeneous subvarieties such that $V_1 \cap V_2$ is trivial. Then $M = M_1 \oplus M_2$ with $\operatorname{V}_H(M_i) = V_i$.*

Suppose now that k is algebraically closed, and fix integers $c \geq 1$ and $a \geq 2$. Define an integer b by

$$b \stackrel{\text{def}}{=} \begin{cases} a/\gcd(a, \operatorname{char} k) & \text{if } \operatorname{char} k > 0 \\ a & \text{if } \operatorname{char} k = 0, \end{cases}$$

and let $q \in k$ be a primitive bth root of unity. Denote by A the *quantum complete intersection* defined by these data, that is, the algebra

$$A \stackrel{\text{def}}{=} k\langle x_1, \ldots, x_c \rangle / (\{x_i^a\}_{i=1}^c, \{x_i x_j - q x_j x_i\}_{i<j}).$$

This local algebra is selfinjective of dimension a^c. Note that when $a = 2$ and $q = -1$, then A is the exterior algebra on a c-dimensional k-vector space.

It follows from [3] that there exists a polynomial subalgebra $H = k[\eta_1, \ldots, \eta_c]$ of $\operatorname{HH}^*(A)$, with each η_i in degree two, such that the H-module $\operatorname{Ext}_A^*(k, k)$ is finitely generated. Thus the finiteness condition from Theorem 1 is satisfied, and so the support varieties with respect to H encode homological information on the A-modules. However, the algebra also has *rank varieties*. Given a c-tuple $\lambda = (\lambda_1, \ldots, \lambda_c) \in k^c$, denote the element $\lambda_1 x_1 + \cdots + \lambda_c x_c \in A$ by u_λ.

Definition. The rank variety of an A-module M is

$$\operatorname{V}_A^r(M) \stackrel{\text{def}}{=} \{0\} \cup \{0 \neq \lambda \in k^c \mid M \text{ is not a projective } k[u_\lambda]\text{-module}\}.$$

The terminology reflects the fact that since $u_\lambda^a = 0$, the algebra $k[u_\lambda]$ is isomorphic to $k[x]/(x^a)$. Hence the condition that M is not $k[u_\lambda]$-projective is equivalent to the condition that the rank of the map $M \xrightarrow{\cdot u_\lambda} M$ be strictly less than $[(a-1)/a] \dim M$.

Thus there are two types of varieties for A-modules. Since we may identify the maximal ideals of H with points in k^c, a natural question arises: is the support variety of a module related to its rank variety? Indeed, for group algebras of elementary abelian p-groups it was conjectured by Carlson (cf. [4]) that the support variety of a module is isomorphic to its rank variety. This was subsequently proved by Avrunin and Scott in [1]. As shown in [2, Theorem 3.6], a similar result holds for our quantum complete intersection A.

Theorem 2. *Let* $k^c \xrightarrow{F} k^c$ *be the map of affine spaces given by* $(\lambda_1, \ldots, \lambda_c) \mapsto (\lambda_1^a, \ldots, \lambda_c^a)$. *Then* $F(\operatorname{V}_A^r(M)) = \operatorname{V}_H(M)$ *for every* A*-module* M.

Corollary 3. For every A-module M, the dimension of the rank variety $\operatorname{V}_A^r(M)$ equals the complexity of M. Moreover, the module is periodic if and only if $\dim \operatorname{V}_A^r(M) = 1$.

References

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