

MODULES WITH REDUCIBLE COMPLEXITY, II

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ABSTRACT. We continue studying the class of modules having reducible complexity over a local ring. In particular, a method is provided for computing an upper bound of the complexity of such a module, in terms of vanishing of certain cohomology modules. We then specialize to complete intersections, which are precisely the rings over which all modules have finite complexity.

1. INTRODUCTION

The notion of complexity of a module was introduced by Alperin and Evens in [AlE], in order to study modular representations of finite groups. A decade later, in [Av1, Av2], Avramov introduced this concept for finitely generated modules over local rings, as a means to distinguish between modules of infinite projective dimension. The complexity of a module measures the growth of its minimal free resolution. For example, a module has complexity zero if and only if its projective dimension is finite, and complexity one precisely when its minimal free resolution is bounded.

For an arbitrary local ring, not much is known about the modules of finite complexity. For example, a characterization of these modules in terms of cohomology does not exist. Even worse, it is unclear whether every local ring has finite finitistic complexity dimension, that is, whether there is a bound on the complexities of the modules having finite complexity. In other words, the status quo for complexity is entirely different from that of projective dimension; there simply does not exist any analogue of the Auslander-Buchsbaum formula. The only class of rings for which these problems are solved are the complete intersections; over such a ring *every* module has finite complexity, and the complexity is bounded by the codimension of the ring.

In this paper we continue studying modules with so-called “reducible complexity”, a concept introduced in [Be2]. In particular, we give a method for computing an upper bound of the complexity of such a module. This is done by looking at the vanishing of cohomology with certain test modules. We then specialize to the case when the ring is a complete intersection, and use the theory of support varieties, introduced in [Av1] and [AvB], both to sharpen our results and to obtain new ones.

2. REDUCIBLE COMPLEXITY

Throughout we let (A, \mathfrak{m}, k) be a local (meaning also commutative Noetherian) ring, and we suppose all modules are finitely generated. For an A -module M with minimal free resolution

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

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the rank of F_n , i.e. the integer $\dim_k \text{Ext}_A^n(M, k)$, is the n th *Betti number* of M , and we denote this by $\beta_n(M)$. The i th *syzygy* of M , denoted $\Omega_A^i(M)$, is the cokernel of the map $F_{i+1} \rightarrow F_i$, and it is unique up to isomorphism. Note that $\Omega_A^0(M) = M$ and that $\beta_n(\Omega_A^i(M)) = \beta_{n+i}(M)$ for all i . The *complexity* of M , denoted $\text{cx } M$, is defined as

$$\text{cx } M = \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq an^{t-1} \text{ for } n \gg 0\}.$$

In general the complexity of a module may be infinite, in fact the rings for which all modules have finite complexity are precisely the complete intersections. From the definition we see that the complexity is zero if and only if the module has finite projective dimension, and that the modules of complexity one are those whose minimal free resolutions are bounded, for example the periodic modules. Moreover, the complexity of M equals that of $\Omega_A^i(M)$ for every $i \geq 0$.

Let N be an A -module, and consider an element $\eta \in \text{Ext}_A^t(M, N)$. By choosing a map $f_\eta: \Omega_A^t(M) \rightarrow N$ representing η , we obtain a commutative pushout diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_A^t(M) & \longrightarrow & F_{t-1} & \longrightarrow & \Omega_A^{t-1}(M) \longrightarrow 0 \\ & & \downarrow f_\eta & & \downarrow & & \parallel \\ 0 & \longrightarrow & N & \longrightarrow & K_\eta & \longrightarrow & \Omega_A^{t-1}(M) \longrightarrow 0 \end{array}$$

with exact rows. Note that the module K_η is independent, up to isomorphism, of the map f_η chosen as a representative for η .

We now recall the definition of modules with reducible complexity. Given A -modules X and Y , we denote the graded A -module $\bigoplus_{i=0}^\infty \text{Ext}_A^i(X, Y)$ by $\text{Ext}_A^*(X, Y)$.

Definition 2.1. *Denote by \mathcal{C}_A the category of all A -modules having finite complexity. The full subcategory $\mathcal{C}_A^r \subseteq \mathcal{C}_A$ consisting of the modules having reducible complexity is defined inductively as follows.*

- (i) *Every module of finite projective dimension belongs to \mathcal{C}_A^r .*
- (ii) *A module $X \in \mathcal{C}_A$ of positive complexity belongs to \mathcal{C}_A^r if there exists a homogeneous element $\eta \in \text{Ext}_A^*(X, X)$ of positive degree such that $\text{cx } K_\eta = \text{cx } X - 1$, $\text{depth } K_\eta = \text{depth } X$ and $K_\eta \in \mathcal{C}_A^r$. In this case, the cohomological element η is said to reduce the complexity of M .*

The notion of reducible complexity was introduced in [Be2], where several (co)homology vanishing results for such modules were given (see also [Be4]). These results were previously known to hold for modules over complete intersections, and so it is not surprising that over such rings all modules have reducible complexity. In fact, by [Be2, Proposition 2.2] every module of finite *complete intersection dimension* has reducible complexity, and given such a module the reducing process decreases the complexity by exactly one. Recall that the module M has finite complete intersection dimension, written $\text{CI-dim}_A M < \infty$, if there exist local rings R and Q and a diagram $A \rightarrow R \leftarrow Q$ of local homomorphisms such that $A \rightarrow R$ is faithfully flat, $R \leftarrow Q$ is surjective with kernel generated by a regular sequence (such a diagram $A \rightarrow R \leftarrow Q$ is called a *quasi-deformation* of A), and $\text{pd}_Q(R \otimes_A M)$ is finite. Such modules were first studied in [AGP], and the concept generalizes that of virtual projective dimension defined in [Av1]. As the name suggests, modules having finite complete intersection dimension to a large extent behave homologically like modules over complete intersections. Indeed, over a complete intersection (S, \mathfrak{n}) every module has finite complete intersection dimension; the completion \widehat{S} of S with respect to the \mathfrak{n} -adic topology is the residue ring of a regular local ring Q modulo an ideal generated by a regular sequence, and so $S \rightarrow \widehat{S} \leftarrow Q$ is a quasi deformation.

It should be commented on the fact that in the original definition of the notion of reducible complexity in [Be2], the reducing process did not necessarily reduce the complexity by exactly one. Namely, if $\eta \in \text{Ext}_A^*(M, M)$ is a cohomological element reducing the complexity of the module M , then the requirement was only that $\text{cx } K_\eta$ be strictly less than $\text{cx } M$, and not necessarily equal to $\text{cx } M - 1$. However, we are unaware of any example where the complexity actually drops by more than one. On the contrary, there is evidence to suggest that this cannot happen. If $\text{CI-dim}_A M$ is finite and

$$0 \rightarrow M \rightarrow K \rightarrow \Omega_A^n(M) \rightarrow 0$$

is an exact sequence, then it is not hard to see that there exists one “joint” quasi deformation $A \rightarrow R \leftarrow Q$ such that all these three modules have finite projective dimension over Q when forwarded to R . In this situation we can apply [Jor, Theorem 1.3], a result originally stated for complete intersections, and obtain a “principal lifting” (of R) over which the complexities drop by exactly one. By altering this procedure, we can show that the complexity of K cannot drop by more than one. Note also that this is trivial if $\text{cx } M \leq 2$.

It should also be noted that the requirement on the depth of the modules involved, i.e. the requirement $\text{depth } K_\eta = \text{depth } X$ in the definition, is not very restrictive. For example, when the ring A is Cohen-Macaulay, this is always the case (see the remark following [Be2, Definition 2.1]).

As mentioned, every A -module of finite complete intersection dimension belongs to \mathcal{C}_A^r . However, the converse is a priori not true, that is, a module in \mathcal{C}_A^r need not have reducible complexity. Moreover, a module having finite complexity need not have free reducible complexity. An example illustrating this was given in [Be2, Section 2], and the following is an example in which the ring is Gorenstein.

Example. Let (A, \mathfrak{m}, k) be the local finite dimensional algebra $k[X_1, \dots, X_5]/\mathfrak{a}$ where $\mathfrak{a} \subset k[X_1, \dots, X_5]$ is the ideal generated by the quadratic forms

$$\begin{aligned} X_1^2, X_2^2, X_5^2, X_3X_4, X_3X_5, X_4X_5, X_1X_4 + X_2X_4 \\ \alpha X_1X_3 + X_2X_3, X_3^2 - X_2X_5 + \alpha X_1X_5, X_4^2 - X_2X_5 + X_1X_5 \end{aligned}$$

for a nonzero element $\alpha \in k$. By [GaP, Proposition 3.1] this ring is Gorenstein, and the complex

$$\dots \rightarrow A^2 \xrightarrow{d_{n+1}} A^2 \xrightarrow{d_n} A^2 \rightarrow \dots$$

with maps given by the matrices $d_n = \begin{pmatrix} x_1 & \alpha^n x_3 + x_4 \\ 0 & x_2 \end{pmatrix}$ is exact. This sequence is therefore a minimal free resolution of the module $M := \text{Im } d_0$, hence this module has complexity one. If the order of α in k is infinite, then M cannot have reducible complexity; if M has reducible complexity, then there exists an exact sequence

$$0 \rightarrow M \rightarrow K \rightarrow \Omega_A^n(M) \rightarrow 0$$

in which K has finite projective dimension. As A is selfinjective, the module K must be free, hence M is a periodic module, a contradiction. Moreover, if the order of α is finite but at least 3, then the argument in [Be2, Section 2, example] shows that M has reducible complexity but not finite complete intersection dimension

This example also shows that, in general, a module of finite *Gorenstein dimension* and finite complexity need not have reducible complexity. Recall that a module X over a local ring R has finite Gorenstein dimension, denoted $\text{G-dim}_R X < \infty$, if there exists an exact sequence

$$0 \rightarrow G_t \rightarrow \dots \rightarrow G_0 \rightarrow X \rightarrow 0$$

of R -modules in which the modules G_i are reflexive and satisfy $\text{Ext}_R^j(G_i, R) = 0 = \text{Ext}_R^j(\text{Hom}_R(G_i, R), R)$ for $j \geq 1$. Every module over a Gorenstein ring has finite

Gorenstein dimension, in fact this property characterizes Gorenstein rings. Using this concept, Gerko introduced in [Ger] the notion of *lower complete intersection dimension*; the module X has finite lower complete intersection dimension, written $\text{CI}_* \text{-dim}_R X < \infty$, if it has finite Gorenstein dimension and finite complexity (and in this case $\text{CI}_* \text{-dim}_R X = \text{G-dim}_R X$). The Gorenstein dimension, lower complete intersection dimension, complete intersection dimension and projective dimension of a module are all related via the inequalities

$$\text{G-dim}_R X \leq \text{CI}_* \text{-dim}_R X \leq \text{CI-dim}_R X \leq \text{pd}_R X.$$

If one of these dimensions happen to be finite, then it is equal to those to its left. Note that the class of modules having reducible complexity and finite Gorenstein dimension lies properly “between” the class of modules having finite lower complete intersection dimension and the class of modules having finite complete intersection dimension.

Given an A -module M , how do we decide whether it has reducible complexity? The following proposition, which does not give a complete answer, shows that at least we can reduce the complexity of M by one when its Ext-algebra is “nice”. Recall that for every A -module N the graded A -module $\text{Ext}_A^*(M, N)$ is a right $\text{Ext}_A^*(M, M)$ -module via the Yoneda product.

Proposition 2.2. *Let M be an A -module, and suppose there exists a commutative Noetherian graded subalgebra H of $\text{Ext}_A^*(M, M)$ over which $\text{Ext}_A^*(M, k)$ is a finitely generated module. Then there exists a homogeneous element $\eta \in \text{Ext}_A^*(M, M)$, of positive degree, such that $\text{cx } K_\eta = \text{cx } M - 1$.*

In order to prove this proposition, we need the following two lemmas, the first of which generalizes [Be1, Proposition 2.1].

Lemma 2.3. *Let H be a commutative graded ring, and let X be a Noetherian graded H -module. Then there exists a homogeneous element $\eta \in H$, of positive degree, such that the multiplication map $X_i \xrightarrow{\eta} X_{i+|\eta|}$ is injective for large i . If H is Noetherian and therefore finitely generated over H_0 by homogeneous elements η_1, \dots, η_t , say, then we may choose η such that $|\eta_i|$ divides $|\eta|$ for some i .*

Proof. Denote by H_+ the positive part $\bigoplus_{i=1}^{\infty} H_i$ of H , and consider the graded submodule

$$L \stackrel{\text{def}}{=} \{x \in X \mid H_+ x = 0\}$$

of X . Since X is Noetherian, this submodule is finitely generated, and by construction it is therefore also finitely generated as an H_0 -module (since $H_0 = H/H_+$). But H_0 lives only in one degree, and consequently $L_i = 0$ for large i . The last part of the proof of [Be1, Proposition 2.1] now applies. \square

The second lemma we need in order to prove Proposition 2.2 is the following version of the Hilbert-Serre Theorem. Its corollary shows that under certain conditions the Poincaré series of a module is a rational function.

Lemma 2.4 (Hilbert-Serre Theorem). *Let K be a field, and let R be a commutative Noetherian graded ring, generated over R_0 by homogeneous elements r_1, \dots, r_t , say. Let V be a graded K - R -bimodule such that $\dim_K V_i < \infty$ for each i , and such that V is a finitely generated R -module. Then the Poincaré series $\sum_{i=0}^{\infty} (\dim_K V_i) x^i$ of V is of the form*

$$\sum_{i=0}^{\infty} (\dim_K V_i) x^i = f(x) / \prod_{i=1}^t (1 - x^{|r_i|}),$$

where $f(x)$ is a polynomial with integer coefficients.

Corollary 2.5. *Let M be an A -module, and suppose there exists a commutative Noetherian graded subalgebra H of $\text{Ext}_A^*(M, M)$ over which $\text{Ext}_A^*(M, k)$ is a finitely generated module. Let H be generated over H_0 by homogeneous elements η_1, \dots, η_t , say. Then the Poincaré series $\sum_{i=0}^{\infty} \beta_i(M)x^i$ of M is of the form*

$$\sum_{i=0}^{\infty} \beta_i(M)x^i = f(x) / \prod_{i=1}^t (1 - x^{|\eta_i|}),$$

where $f(x)$ is a polynomial with integer coefficients. Moreover, the complexity of M is finite and equal to the order of the pole at $x = 1$ of the Poincaré series.

Proof. The rationality of the Poincaré series follows from the Hilbert-Serre Theorem, since $\text{Ext}_A^*(M, k)$ is a k - H -bimodule and $\beta_i(M) = \dim_k \text{Ext}_A^i(M, k)$. The last statement follows from [Ben, Proposition 5.3.2]. \square

We are now ready to prove Proposition 2.2.

Proof of Proposition 2.2. Since H is Noetherian, it is generated over H_0 by homogeneous elements η_1, \dots, η_t , say. By Lemma 2.3, there exists a homogeneous element $\eta \in \text{Ext}_A^*(M, M)$, of positive degree, such that scalar multiplication

$$\text{Ext}_A^i(M, k) \xrightarrow{\eta} \text{Ext}_A^{i+|\eta|}(M, k)$$

is injective for large i . Moreover, we may assume that the degree of one η_1, \dots, η_t divides that of η . Consider the short exact sequence

$$0 \rightarrow M \rightarrow K_\eta \rightarrow \Omega_A^{|\eta|-1}(M) \rightarrow 0$$

induced by this element. Applying $\text{Hom}_A(-, k)$ to this sequence, we obtain a long exact sequence

$$\dots \rightarrow \text{Ext}_A^i(K_\eta, k) \rightarrow \text{Ext}_A^i(M, k) \xrightarrow{\partial_\eta} \text{Ext}_A^{i+|\eta|}(M, k) \rightarrow \text{Ext}_A^{i+1}(K_\eta, k) \rightarrow \dots$$

in which the connecting homomorphism $\text{Ext}_A^i(M, k) \xrightarrow{\partial_\eta} \text{Ext}_A^{i+|\eta|}(M, k)$ is just scalar multiplication with $(-1)^i \eta$ (cf. [Mac, Theorem III.9.1]). This map is injective for large i , hence there exists an integer i_0 such that

$$\beta_{i+1}(K_\eta) = \beta_{i+|\eta|}(M) - \beta_i(M)$$

for $i \geq i_0$.

From Corollary 2.5 we know that the the Poincaré series $\sum_{i=0}^{\infty} \beta_i(M)x^i$ of M is of the form

$$\sum_{i=0}^{\infty} \beta_i(M)x^i = f(x) / \prod_{i=1}^t (1 - x^{|\eta_i|}),$$

where $f(x)$ is a polynomial with integer coefficients. Denote this rational function by $P(M, x)$, and denote the Poincaré series $\sum_{i=0}^{\infty} \beta_i(K_\eta)x^i$ of K_η by $P(K_\eta, x)$. From the equality

$$\sum_{i=i_0}^{\infty} \beta_{i+1}(K_\eta)x^{i+1} = \sum_{i=i_0}^{\infty} \beta_{i+|\eta|}(M)x^{i+1} - \sum_{i=i_0}^{\infty} \beta_i(M)x^{i+1}$$

we obtain the equality

$$P(K_\eta, x) = \frac{(1 - x^{|\eta|})P(M, x)}{x^{|\eta|-1}} + \frac{g(x)}{x^{|\eta|-1}},$$

where $g(x)$ is a polynomial with integer coefficients. Now since one of $|\eta_1|, \dots, |\eta_t|$ divides $|\eta|$, we see that the order of the pole of $P(K_\eta, x)$ at $x = 1$ is exactly one less than that of $P(M, x)$. Then from [Ben, Proposition 5.3.2] we see that $\text{cx } K_\eta = \text{cx } M - 1$. \square

3. COMPLEXITY TESTING

In this section we introduce a method for computing an upper bound for the complexity of a given module in \mathcal{C}_A^r . We start with a key result from [Be2], showing that the modules in \mathcal{C}_A^r having infinite projective dimension also have higher self extensions.

Proposition 3.1. [Be2, Corollary 3.2] *If M belongs to \mathcal{C}_A^r , then*

$$\text{pd } M = \sup\{i \mid \text{Ext}_A^i(M, M) \neq 0\}.$$

Thus every module in \mathcal{C}_A^r with infinite projective dimension has higher self-extensions. For a given natural number t , denote by $\mathcal{C}_A^r(t)$ the full subcategory

$$\mathcal{C}_A^r(t) \stackrel{\text{def}}{=} \{X \in \mathcal{C}_A^r \mid \text{cx } X = t\}$$

of \mathcal{C}_A^r consisting of the modules of complexity t . The following result shows that this subcategory serves as a ‘‘complexity test category’’, in the sense that if a module has no higher extensions with the modules in $\mathcal{C}_A^r(t)$, then its complexity is strictly less than t .

Theorem 3.2. *Let M be a module belonging to \mathcal{C}_A^r and t a natural number. If $\text{Ext}_A^i(M, N) = 0$ for every $N \in \mathcal{C}_A^r(t)$ and $i \gg 0$, then $\text{cx } M < t$.*

Proof. We show by induction that if $\text{cx } M \geq t$, then there is a module $N \in \mathcal{C}_A^r(t)$ with the property that $\text{Ext}_A^i(M, N)$ does not vanish for all $i \gg 0$. If the complexity of M is t , then by Proposition 3.1 we may take N to be M itself, so suppose that $\text{cx } M > t$. Choose a cohomological homogeneous element $\eta \in \text{Ext}_A^*(M, M)$ of positive degree reducing the complexity. In the corresponding exact sequence

$$0 \rightarrow M \rightarrow K_\eta \rightarrow \Omega_A^{|\eta|-1}(M) \rightarrow 0,$$

the module K_η also belongs to \mathcal{C}_A^r and has complexity one less than that of M , hence by induction there is a module $N \in \mathcal{C}_A^r(t)$ such that $\text{Ext}_A^i(K_\eta, N)$ does not vanish for all $i \gg 0$. From the long exact sequence

$$\dots \rightarrow \text{Ext}_A^{i+|\eta|-1}(M, N) \rightarrow \text{Ext}_A^i(K_\eta, N) \rightarrow \text{Ext}_A^i(M, N) \rightarrow \text{Ext}_A^{i+|\eta|}(M, N) \rightarrow \dots$$

resulting from η , we see that $\text{Ext}_A^i(M, N)$ cannot vanish for all $i \gg 0$. \square

In particular, we can use the category $\mathcal{C}_A^r(1)$ to decide whether a given module in \mathcal{C}_A^r has finite projective dimension. We record this fact in the following corollary.

Corollary 3.3. *A module $M \in \mathcal{C}_A^r$ has finite projective dimension if and only if $\text{Ext}_A^i(M, N) = 0$ for every $N \in \mathcal{C}_A^r(1)$ and $i \gg 0$.*

Remark. Let \mathcal{C}_A^{ci} denote the category of all A -modules of finite complete intersection dimension, and for each natural number t define the category

$$\mathcal{C}_A^{ci}(t) \stackrel{\text{def}}{=} \{X \in \mathcal{C}_A^{ci} \mid \text{cx } X = t\}.$$

Then Theorem 3.2 and Corollary 3.3 remain true if we replace \mathcal{C}_A^r and $\mathcal{C}_A^r(t)$ by \mathcal{C}_A^{ci} and $\mathcal{C}_A^{ci}(t)$, respectively. That is, when the module we are considering has finite complete intersection dimension, then we need only use modules of finite complete intersection dimension as test modules.

When the ring is Gorenstein, then it follows from [Be2, Theorem 3.5] that symmetry holds for the vanishing of cohomology between modules of reducible complexity. We therefore have the following symmetric version of Theorem 3.2.

Corollary 3.4. *Suppose A is Gorenstein, let M be a module belonging to \mathcal{C}_A^r , and let t be a natural number. If $\text{Ext}_A^i(N, M) = 0$ for every $N \in \mathcal{C}_A^r(t)$ and $i \gg 0$, then $\text{cx } M < t$.*

We now turn to the setting in which *every* A -module has reducible complexity. For the remainder of this section, we assume A is a complete intersection, i.e. the \mathfrak{m} -adic completion \widehat{A} of A is the residue ring of a regular local ring modulo a regular sequence. For such rings, Avramov and Buchweitz introduced in [Av1] and [AvB] a theory of cohomological support varieties, and they showed that this theory is similar to that of the cohomological support varieties for group algebras. As we will implicitly use this powerful theory in the results to come, we recall now the definitions (details can be found in [Av1, Section 1] and [AvB, Section 2]).

Denote by c the codimension of A , that is, the integer $\dim_k(\mathfrak{m}/\mathfrak{m}^2) - \dim A$, and by χ the sequence χ_1, \dots, χ_c consisting of the c commuting Eisenbud operators of cohomological degree two. For every \widehat{A} -module X there is a homomorphism

$$\widehat{A}[\chi] \xrightarrow{\phi_X} \text{Ext}_{\widehat{A}}^*(X, X)$$

of graded rings, and via this homomorphism $\text{Ext}_{\widehat{A}}^*(X, Y)$ is finitely generated over $\widehat{A}[\chi]$ for every \widehat{A} -module Y . Denote by H the polynomial ring $k[\chi]$, and by $E(X, Y)$ the graded space $\text{Ext}_{\widehat{A}}^*(X, Y) \otimes_{\widehat{A}} k$. The above homomorphism ϕ_X , together with the canonical isomorphism $H \simeq \widehat{A}[\chi] \otimes_{\widehat{A}} k$, induce a homomorphism $H \rightarrow E(X, X)$ of graded rings, under which $E(X, Y)$ is a finitely generated H -module. Now let M be an A -module, and denote by \widehat{M} its \mathfrak{m} -adic completion $\widehat{A} \otimes_A M$. The support variety $V(M)$ of M is the algebraic set

$$V(M) \stackrel{\text{def}}{=} \{\alpha \in \tilde{k}^c \mid f(\alpha) = 0 \text{ for all } f \in \text{Ann}_H E(\widehat{M}, \widehat{M})\},$$

where \tilde{k} is the algebraic closure of k . Finally, for an ideal $\mathfrak{a} \subseteq H$ we define the variety $V_H(\mathfrak{a}) \subseteq \tilde{k}^c$ to be the zero set of \mathfrak{a} .

As mentioned above, this theory shares many properties with the theory of cohomological support varieties for modules over group algebras of finite groups. For instance, the dimension of the variety of a module equals the complexity of the module, in particular the variety is trivial if and only if the module has finite projective dimension. The following complexity test result relies on [Be3, Corollary 2.3], which says that every homogeneous algebraic subset of \tilde{k}^c is realizable as the support variety of some A -module.

Proposition 3.5. *Let M be an A -module, let $\eta_1, \dots, \eta_t \in H$ be homogeneous elements of positive degrees, and choose an A -module $T_{\eta_1, \dots, \eta_t}$ with the property that $V(T_{\eta_1, \dots, \eta_t}) = V_H(\eta_1, \dots, \eta_t)$. If $\text{Ext}_A^i(M, T_{\eta_1, \dots, \eta_t}) = 0$ for $i \gg 0$, then $\text{cx } M \leq t$.*

Proof. Denote the ideal $\text{Ann}_H E(\widehat{M}, \widehat{M}) \subseteq H$ by \mathfrak{a} . If $\text{Ext}_A^i(M, T_{\eta_1, \dots, \eta_t}) = 0$ for $i \gg 0$, then from [AvB, Theorem 5.6] we obtain

$$\{0\} = V(M) \cap V(T_{\eta_1, \dots, \eta_t}) = V_H(\mathfrak{a}) \cap V_H(\eta_1, \dots, \eta_t) = V_H(\mathfrak{a} + (\eta_1, \dots, \eta_t)),$$

hence the ring $H/(\mathfrak{a} + (\eta_1, \dots, \eta_t))$ is zero dimensional. But then the dimension of the ring H/\mathfrak{a} is at most t , i.e. $\text{cx } M \leq t$. \square

We illustrate this last result with an example.

Example. Let k be a field and Q the formal power series ring $k[[x_1, \dots, x_c]]$ in c variables. For each $1 \leq i \leq c$, let $n_i \geq 2$ be an integer, let $\mathfrak{a} \subseteq Q$ be the ideal generated by the regular sequence $x_1^{n_1}, \dots, x_c^{n_c}$, and denote by A the complete intersection Q/\mathfrak{a} . For each $1 \leq i \leq c$ we shall construct an A -module whose support variety equals $V_H(\chi_i)$, by adopting the techniques used in [SnS, Section 7] to give an interpretation of the Eisenbud operators.

Consider the exact sequence

$$0 \rightarrow \mathfrak{m}_Q \rightarrow Q \rightarrow k \rightarrow 0$$

of Q -modules. Applying $A \otimes_Q -$ to this sequence gives the four term exact sequence

$$(\dagger) \quad 0 \rightarrow \mathrm{Tor}_1^Q(A, k) \rightarrow \mathfrak{m}_Q / \mathfrak{a} \mathfrak{m}_Q \rightarrow A \rightarrow k \rightarrow 0$$

of A -modules. Consider the first term in this sequence. By tensoring the exact sequence

$$0 \rightarrow \mathfrak{a} \rightarrow Q \rightarrow A \rightarrow 0$$

over Q with k , we obtain the exact sequence

$$0 \rightarrow \mathrm{Tor}_1^Q(A, k) \rightarrow \mathfrak{a} \otimes_Q k \xrightarrow{g} Q \otimes_Q k \rightarrow A \otimes_Q k \rightarrow 0,$$

in which the map g must be the zero map since $\mathfrak{a}k = 0$. This gives isomorphisms

$$\mathrm{Tor}_1^Q(A, k) \simeq \mathfrak{a} \otimes_Q k \simeq \mathfrak{a} \otimes_Q (A \otimes_A k) \simeq \mathfrak{a} / \mathfrak{a}^2 \otimes_A k$$

of A -modules. Since \mathfrak{a} is generated by a regular sequence of length c , the A -module $\mathfrak{a} / \mathfrak{a}^2$ is free of rank c , and therefore $\mathrm{Tor}_1^Q(A, k)$ is isomorphic to k^c . We may now rewrite the four term exact sequence (\dagger) as

$$0 \rightarrow k^c \xrightarrow{f} \mathfrak{m}_Q / \mathfrak{a} \mathfrak{m}_Q \rightarrow A \rightarrow k \rightarrow 0,$$

and it is not hard to show that the map f is defined by

$$(\alpha_1, \dots, \alpha_c) \mapsto \sum \alpha_i x_i^{n_i} + \mathfrak{a} \mathfrak{m}_Q.$$

The image of the Eisenbud operator χ_j under the homomorphism $\widehat{A}[\chi] \xrightarrow{\phi_k} \mathrm{Ext}_A^*(k, k)$ is the bottom row in the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k^c & \xrightarrow{f} & \mathfrak{m}_Q / \mathfrak{a} \mathfrak{m}_Q & \longrightarrow & A & \longrightarrow & k & \longrightarrow & 0 \\ & & \downarrow \pi_j & & \downarrow & & \parallel & & \parallel & & \\ 0 & \longrightarrow & k & \longrightarrow & K_{\chi_j} & \longrightarrow & A & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

of A -modules, in which the map π_j is projection onto the j th summand. The pushout module K_{χ_j} can be described explicitly as

$$K_{\chi_j} = \frac{k \oplus \mathfrak{m}_Q / \mathfrak{a} \mathfrak{m}_Q}{\{(\alpha_j, -\sum \alpha_i x_i^{n_i} + \mathfrak{a} \mathfrak{m}_Q) \mid (\alpha_1, \dots, \alpha_c) \in k^c\}},$$

and by [Be3, Theorem 2.2] its support variety is given by $V(K_{\chi_j}) = V(k) \cap V_H(\chi_j)$. But the variety of k is the whole space, hence the equality $V(K_{\chi_j}) = V_H(\chi_j)$. Thus by Proposition 3.5 the A -module K_{χ_j} is a test module for finding modules with bounded projective resolutions; if M is an A -module such that $\mathrm{Ext}_A^i(M, K_{\chi_j}) = 0$ for $i \gg 0$, then $\mathrm{cx} M \leq 1$.

Before proving the final result, we need a lemma showing that every maximal Cohen-Macaulay module over a complete intersection has reducible complexity by a cohomological element of degree two. This improves [Be4, Lemma 2.1(i)], which states that such a cohomological element exists after passing to some suitable faithfully flat extension of the ring.

Lemma 3.6. *If M is a maximal Cohen-Macaulay A -module of infinite projective dimension, then there exists an element $\eta \in \mathrm{Ext}_A^2(M, M)$ reducing its complexity.*

Proof. Since the dimension of $V(M)$ is nonzero, the radical $\sqrt{\mathrm{Ann}_H E(\widehat{M}, \widehat{M})}$ of $\mathrm{Ann}_H E(\widehat{M}, \widehat{M})$ is properly contained in the graded maximal ideal of H . Therefore one of the Eisenbud operators, say χ_j , is not contained in $\sqrt{\mathrm{Ann}_H E(\widehat{M}, \widehat{M})}$. We now follow the arguments given prior to [Be3, Corollary 2.3]. Viewing χ_j

as an element of $\widehat{A}[\chi]$, we can apply the homomorphism $\phi_{\widehat{M}}$ and obtain the element $\phi_{\widehat{M}}(\chi_j) \otimes 1$ in $\text{Ext}_{\widehat{A}}^2(\widehat{M}, \widehat{M}) \otimes_{\widehat{A}} k$. Now $\text{Ext}_{\widehat{A}}^2(\widehat{M}, \widehat{M})$ is isomorphic to $\text{Ext}_A^2(M, M) \otimes_A \widehat{A}$, and there is an isomorphism

$$\text{Ext}_A^2(M, M) \otimes_A k \xrightarrow{\sim} \text{Ext}_{\widehat{A}}^2(\widehat{M}, \widehat{M}) \otimes_{\widehat{A}} k$$

mapping an element $\theta \otimes 1 \in \text{Ext}_A^2(M, M) \otimes_A k$ to $\widehat{\theta} \otimes 1$. Therefore there exists an element $\eta \in \text{Ext}_A^2(M, M)$ such that $\widehat{\eta} \otimes 1$ equals $\phi_{\widehat{M}}(\chi_j) \otimes 1$ in $\text{Ext}_{\widehat{A}}^2(\widehat{M}, \widehat{M}) \otimes_{\widehat{A}} k$. If the exact sequence

$$0 \rightarrow M \rightarrow K_\eta \rightarrow \Omega_A^1(M) \rightarrow 0$$

corresponds to η , then its completion

$$0 \rightarrow \widehat{M} \rightarrow \widehat{K}_\eta \rightarrow \Omega_A^1(\widehat{M}) \rightarrow 0$$

corresponds to $\widehat{\eta}$, and so from [Be3, Theorem 2.2] we see that

$$V(K_\eta) = V(M) \cap V_H(\chi_j).$$

Since χ_j was chosen so that it ‘‘cuts down’’ the variety of M , we must have $\dim V(K_\eta) = \dim V(M) - 1$, i.e. $\text{cx } K_\eta = \text{cx } M - 1$. \square

We have now arrived at the final result, which improves Theorem 3.2 when the ring is a complete intersection. Namely, for such rings it suffices to check the vanishing of finitely many cohomology groups ‘‘separated’’ by an odd number. The number of cohomology groups we need to check depends on the complexity value we are testing. Recall that we have denoted the codimension of the complete intersection A by c .

Theorem 3.7. *Let M be an A -module and $t \in \{1, \dots, c\}$ an integer. If for every A -module N of complexity t there is an odd number q such that*

$$\text{Ext}_A^n(M, N) = \text{Ext}_A^{n+q}(M, N) = \dots = \text{Ext}_A^{n+(c-t)q}(M, N) = 0$$

for some even number $n > \dim A - \text{depth } M$, then $\text{cx } M < t$.

Proof. Since $\text{cx } M = \text{cx } \Omega_A^{\dim A - \text{depth } M}(M)$, we may without loss of generality assume that M is maximal Cohen-Macaulay and that $n > 0$. We prove by induction that if $\text{cx } M \geq t$, then for any odd number q and any even integer $n > 0$, the groups

$$\text{Ext}_A^n(M, N), \text{Ext}_A^{n+q}(M, N), \dots, \text{Ext}_A^{n+(c-t)q}(M, N)$$

cannot all vanish for every module N of complexity t . When the complexity of M is t , take N to be M itself. In this case it follows from [AvB, Theorem 4.2] that $\text{Ext}_A^n(M, N)$ is nonzero, because $t \geq 1$. Now assume $\text{cx } M > t$, and write q as $q = 2s - 1$ where $s \geq 1$ is an integer. By Lemma 3.6 there is an element $\eta \in \text{Ext}_A^2(M, M)$ reducing the complexity of M , and it follows from [Be2, Proposition 2.4(i)] that the element $\eta^s \in \text{Ext}_A^{2s}(M, M)$ also reduces the complexity. The latter element corresponds to an exact sequence

$$0 \rightarrow M \rightarrow K \rightarrow \Omega_A^q(M) \rightarrow 0,$$

in which the complexity of K is one less than that of M . By induction there exists a module N , of complexity t , such that the groups

$$\text{Ext}_A^n(K, N), \text{Ext}_A^{n+q}(K, N), \dots, \text{Ext}_A^{n+(c-t)q}(K, N)$$

do not all vanish. Then from the exact sequence we see that the groups

$$\text{Ext}_A^n(M, N), \text{Ext}_A^{n+q}(M, N), \dots, \text{Ext}_A^{n+(c-t)q}(M, N)$$

cannot possibly all vanish. Since the complexity of any A -module is at most c , the proof is complete. \square

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