

RELATIVE SUPPORT VARIETIES

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ABSTRACT. We define relative support varieties with respect to some fixed module over a finite dimensional algebra. These varieties share many of the standard properties of classical support varieties. Moreover, when introducing finite generation conditions on cohomology, we show that relative support varieties contain homological information on the modules involved. As an application, we provide a new criterion for a selfinjective algebra to be of wild representation type.

1. INTRODUCTION

Support varieties for modules over a given algebra are defined in terms of the maximal ideal spectrum of some commutative graded ring of cohomology operators, operators which act centrally on the cohomology groups of the algebra. For group algebras of finite groups, or, more generally, for finite dimensional cocommutative Hopf algebras, this role is played by the cohomology ring of the algebra (cf. [4], [7], [12], [14]). For commutative local complete intersections, one uses the polynomial ring of Eisenbud operators (cf. [2], [3]). In all these cases, the ring of cohomology operators is Noetherian, and all the cohomology groups of the algebra are finitely generated as modules. Consequently, the theory of support varieties over these rings is very powerful, in that the variety of a module contains a lot of homological information on the module itself.

As shown in [18], for a finite dimensional algebra, the Hochschild cohomology ring, with its maximal ideal spectrum, is a natural candidate as a ring of central cohomology operators. However, in general this ring is not Noetherian, and the cohomology groups of the algebra are not always finitely generated modules. But, as shown in [10], when the Hochschild cohomology ring *is* Noetherian and all the cohomology groups are finitely generated, then one obtains a support variety theory very much like in the classical cases. It is therefore important to establish which finite dimensional algebras have “nice” Hochschild cohomology rings. For quantum complete intersections, this has been solved (cf. [11], [6]).

In this paper, we define *relative* support varieties with respect to a fixed module. These are defined in terms of the maximal ideal spectrum of some commutative graded subalgebra of the Ext-algebra of the module. As one would expect, these varieties share many of the same properties of “ordinary” support varieties, such as the standard behavior on exact sequences etc. Moreover, when we introduce finite generation conditions, then the relative support varieties contain homological information on the modules involved, just as in the classical case.

As an application, we provide a new criterion for a finite dimensional selfinjective algebra to be of wild representation type. Namely, we show that if there exists a module whose Ext-algebra is “large” enough, Noetherian and finitely generated as a module over its center, then the algebra is wild. This generalizes Farnsteiner’s

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theorem, which states that the complexity of every module of a tame block of a finite group scheme is at most two (cf. [13]).

2. RELATIVE SUPPORT VARIETIES

Throughout this paper, we let k be a field and Λ a finite dimensional k -algebra. We denote by $\text{mod } \Lambda$ the category of finitely generated left Λ -modules, and we fix a module $M \in \text{mod } \Lambda$ whose higher self-extensions do not all vanish. Whenever we deal with Λ -modules, we assume they belong to $\text{mod } \Lambda$. Finally, for two Λ -modules X and Y , we denote by $\text{Ext}_\Lambda^*(X, Y)$ the direct sum $\bigoplus_{i=0}^\infty \text{Ext}_\Lambda^i(X, Y)$.

Consider the Ext-algebra $\text{Ext}_\Lambda^*(M, M)$ of M , in which multiplication is given by the Yoneda product. Then for any Λ -module N , the graded k -vector space $\text{Ext}_\Lambda^*(M, N)$ is a graded right $\text{Ext}_\Lambda^*(M, M)$ -module, whereas $\text{Ext}_\Lambda^*(N, M)$ is a graded left $\text{Ext}_\Lambda^*(M, M)$ -module. Moreover, a Λ -homomorphism $N_1 \xrightarrow{f} N_2$ induces homomorphisms

$$\begin{array}{ccc} \text{Ext}_\Lambda^*(M, N_1) & \xrightarrow{f_*} & \text{Ext}_\Lambda^*(M, N_2) \\ \text{Ext}_\Lambda^*(N_2, M) & \xrightarrow{f^*} & \text{Ext}_\Lambda^*(N_1, M) \end{array}$$

of right and left $\text{Ext}_\Lambda^*(M, M)$ -modules. The homomorphism f_* is given as follows: given a homogeneous element

$$\eta: 0 \rightarrow N_1 \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow M \rightarrow 0$$

in $\text{Ext}_\Lambda^*(M, N_1)$, the element $f_*(\eta)$ is the lower exact sequence in the diagram

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & N_1 & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \parallel & & & & \parallel & & \parallel & & \\ 0 & \longrightarrow & N_2 & \longrightarrow & K & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

in which the module K is a pushout. Similarly, the homomorphism f^* is induced by pullback along f . It follows immediately that f_* and f^* are well defined homomorphisms of right and left $\text{Ext}_\Lambda^*(M, M)$ -modules, respectively.

The relative support varieties are defined with respect to some commutative graded subalgebra of $\text{Ext}_\Lambda^*(M, M)$, and therefore we now fix such a subalgebra.

Assumption. Fix a commutative graded subalgebra $H \subseteq \text{Ext}_\Lambda^*(M, M)$ such that H_0 is a local ring.

As mentioned, the relative support varieties to be defined are defined with respect to this graded subalgebra H . The assumption that H_0 is local is made in order to get a nice characterization of the trivial varieties. This assumption is not very restrictive. For example, when we introduce finiteness assumptions later, then we may actually take H to be a polynomial ring over k , so that H_0 is just k itself. Moreover, the following result shows that when M is an indecomposable module, then H_0 is automatically a local ring.

Lemma 2.1. *If M is indecomposable, then H_0 is automatically a local ring.*

Proof. Since H_0 is a finite dimensional commutative k -algebra, the factor algebra $H_0/\text{rad } H_0$ is a product $K_1 \times \cdots \times K_t$ of fields. If $t \geq 2$, then this factor algebra contains nontrivial idempotents, and these lift to H_0 . But H_0 , being a subalgebra of $\text{End}_\Lambda(M)$, cannot contain any nontrivial idempotent, hence $t = 1$. Therefore the radical of H_0 is a maximal ideal. \square

Since we have assumed that H_0 is a local ring, the graded ideal $\text{rad } H_0 \oplus H_1 \oplus \cdots$ is maximal in H (and it is the only maximal graded ideal). We denote this ideal by $\mathfrak{m}_{\text{gr}}(H)$.

We now define a relative support variety theory for Λ -modules, in which the commutative graded ring H is the coordinate ring. Given a Λ -module N , denote by $\text{Ann}_H^i N$ the annihilator of $\text{Ext}_\Lambda^*(M, N)$ in H , and by $\text{Ann}_H^p N$ the annihilator of $\text{Ext}_\Lambda^*(N, M)$ in H . As the annihilator of any graded module over any graded ring is graded, the ideals $\text{Ann}_H^i N$ and $\text{Ann}_H^p N$ are graded ideals of H . We define the injective and projective *support varieties* of N with respect to H as

$$\begin{aligned} V_H^i(N) &\stackrel{\text{def}}{=} \{ \mathfrak{m} \in \text{MaxSpec } H \mid \text{Ann}_H^i N \subseteq \mathfrak{m} \}, \\ V_H^p(N) &\stackrel{\text{def}}{=} \{ \mathfrak{m} \in \text{MaxSpec } H \mid \text{Ann}_H^p N \subseteq \mathfrak{m} \}, \end{aligned}$$

respectively, where $\text{MaxSpec } H$ denotes the set of maximal ideals of H . Note that $\text{Ann}_H^i N$ and $\text{Ann}_H^p N$ are contained in $\mathfrak{m}_{\text{gr}}(H)$, hence $\mathfrak{m}_{\text{gr}}(H)$ is trivially a point in both $V_H^i(N)$ and $V_H^p(N)$. We call a variety *trivial* if it only contains this point.

In the following result we record some elementary facts on relative varieties. Whenever we write $V_H^*(N)$ or $\text{Ann}_H^* N$ and make a statement, it is to be understood that the statement holds in both the injective and projective cases. Furthermore, denote by M^\perp the category of all Λ -modules X such that $\text{Ext}_\Lambda^n(M, X) = 0$ for $n \gg 0$, and by ${}^\perp M$ the category of all Λ -modules Y such that $\text{Ext}_\Lambda^n(Y, M) = 0$ for $n \gg 0$.

Proposition 2.2. *For Λ -modules N, N_1, N_2, N_3 , the following hold:*

- (i) $V_H^*(M) = \text{MaxSpec } H$.
- (ii) *If $N \in M^\perp$, then $V_H^i(N)$ is trivial. In particular, this holds if the injective dimension of N is finite.*
- (iii) *If $N \in {}^\perp M$, then $V_H^p(N)$ is trivial. In particular, this holds if the projective dimension of N is finite.*
- (iv) *For any exact sequence*

$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0,$$

the inclusion $V_H^(N_u) \subseteq V_H^*(N_v) \cup V_H^*(N_w)$ holds whenever $\{u, v, w\} = \{1, 2, 3\}$.*

- (v) *If $N = N_1 \oplus N_2$, then $V_H^*(N) = V_H^*(N_1) \cup V_H^*(N_2)$.*

Proof. Since H is a subalgebra of $\text{Ext}_\Lambda^*(M, M)$, no nonzero element of H can annihilate $\text{Ext}_\Lambda^*(M, M)$. Therefore $\text{Ann}_H^* M = 0$, and this shows (i).

To prove (ii), note that if $\text{Ext}_\Lambda^n(M, N) = 0$ for $n \gg 0$ and η is a homogeneous element in H of positive degree, then some power of η belongs to $\text{Ann}_H^i N$. Moreover, if θ is any element of $\text{rad } H_0$, then it is nilpotent, and therefore some power of θ also belongs to $\text{Ann}_H^i N$. Consequently $V_H^i(N) = \{\mathfrak{m}_{\text{gr}}(H)\}$. This proves (ii), and the proof of (iii) is similar.

As for (iv), we prove only the inclusion $V_H^i(N_3) \subseteq V_H^i(N_1) \cup V_H^i(N_2)$; the other inclusions are proved analogously. The given short exact sequence induces an exact sequence

$$\text{Ext}_\Lambda^*(M, N_2) \rightarrow \text{Ext}_\Lambda^*(M, N_3) \rightarrow \text{Ext}_\Lambda^{*+1}(M, N_1)$$

of right $\text{Ext}_\Lambda^*(M, M)$ -modules, from which we obtain $\text{Ann}_H^i N_2 \cdot \text{Ann}_H^i N_1 \subseteq \text{Ann}_H^i N_3$. The inclusion $V_H^i(N_3) \subseteq V_H^i(N_1) \cup V_H^i(N_2)$ now follows.

The proof of (v) is straightforward. \square

By combining properties (ii), (iii) and (iv) in Proposition 2.2, we see that injective varieties are invariant under cosyzygies, whereas projective varieties are invariant

under syzygies. We record these facts in the following slightly more general result, which concludes this section.

Corollary 2.3. *Let N be a Λ -module, and let*

$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$$

be an exact sequence in $\text{mod } \Lambda$.

- (i) *If $N_2 \in M^\perp$, then $V_M^i(N_1) = V_M^i(N_3)$. In particular, the injective variety of N equals that of $\Omega_\Lambda^{-1}(N)$.*
- (ii) *If $N_2 \in {}^\perp M$, then $V_M^p(N_1) = V_M^p(N_3)$. In particular, the projective variety of N equals that of $\Omega_\Lambda^1(N)$.*

3. FINITE GENERATION

The reason why the theories of support varieties for group rings, cocommutative Hopf algebras and complete intersections are all very powerful, is the existence of a central commutative Noetherian ring over which all the cohomology groups are finitely generated (cf. [2], [3], [4], [7], [12], [14]). As shown in [10], a similar theory is obtained for support varieties defined in terms of the Hochschild cohomology ring, when one *assumes* the existence of such a commutative ring. Motivated by this, we now make the following assumption on the fixed subalgebra H of $\text{Ext}_\Lambda^*(M, M)$.

Assumption. The ring H is Noetherian.

A priori, the algebra H is just some *unknown* graded subalgebra of $\text{Ext}_\Lambda^*(M, M)$, and this is of course not satisfactory if we want to do real computations. However, the following result shows that when H is a subalgebra of the center $Z(M)$ of $\text{Ext}_\Lambda^*(M, M)$, and we require $\text{Ext}_\Lambda^*(M, M)$ to be a finitely generated H -module, then we may take H to be $Z(M)$ itself. Note that $Z(M)$ is a graded algebra. Indeed, suppose η is an element of $Z(M)$, and write $\eta = \eta_0 + \cdots + \eta_n$, where η_i is an element of $\text{Ext}_\Lambda^i(M, M)$ for each i . Let θ be any homogeneous element of $\text{Ext}_\Lambda^*(M, M)$. Then since $\eta\theta = \theta\eta$, we see that each η_i must commute with θ . Therefore each η_i belongs to $Z(M)$, and this shows that $Z(M)$ is a graded algebra.

Proposition 3.1. *The following are equivalent.*

- (i) *There exists a commutative Noetherian graded subalgebra $R \subseteq Z(M)$ over which $\text{Ext}_\Lambda^*(M, M)$ is a finitely generated module.*
- (ii) *The ring $Z(M)$ is Noetherian, and $\text{Ext}_\Lambda^*(M, M)$ is a finitely generated $Z(M)$ -module.*
- (iii) *The ring $\text{Ext}_\Lambda^*(M, M)$ is Noetherian and a finitely generated $Z(M)$ -module.*

Proof. The implication (ii) \Rightarrow (i) is obvious. Suppose (i) holds, and let G be an algebra “lying between” R and $\text{Ext}_\Lambda^*(M, M)$, i.e. $R \subseteq G \subseteq \text{Ext}_\Lambda^*(M, M)$. Then $\text{Ext}_\Lambda^*(M, M)$ must be a finitely generated G -module. Moreover, since R is Noetherian and $\text{Ext}_\Lambda^*(M, M)$ is a finitely generated R -module, we see that $\text{Ext}_\Lambda^*(M, M)$ is a Noetherian ring. This shows the implication (i) \Rightarrow (iii). Finally, the implication (iii) \Rightarrow (ii) is [1, Theorem 1]. \square

As mentioned in the previous section, the assumption that H_0 is a local ring is superfluous once we have introduced finiteness conditions. Namely, the following result shows that we may take H to be a polynomial ring over k , so that H_0 is just k itself. Recall first that if V is a graded k -vector space of finite type (i.e. $\dim_k V_i < \infty$ for all i), then the *rate of growth* of V , denoted $\gamma(V)$, is defined as

$$\gamma(V) \stackrel{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \dim_k V_n \leq an^{t-1} \text{ for } n \gg 0\}.$$

Proposition 3.2. *Let N be a Λ -module, and suppose $\text{Ext}_\Lambda^*(M, N)$ (respectively, $\text{Ext}_\Lambda^*(N, M)$) is a finitely generated H -module. Then there exists a polynomial ring $k[x_1, \dots, x_c] \subseteq H$, with $c = \gamma(H)$, such that H and $\text{Ext}_\Lambda^*(M, N)$ (respectively, $\text{Ext}_\Lambda^*(N, M)$) are finitely generated $k[x_1, \dots, x_c]$ -modules.*

Proof. Follows from the Noether normalization lemma. \square

In the following result we characterize precisely when *all* the cohomology modules are finitely generated over H .

Proposition 3.3. *Consider the following conditions.*

- (i) *For all $N \in \text{mod } \Lambda$, the H -module $\text{Ext}_\Lambda^*(M, N)$ is finitely generated.*
- (ii) *The H -module $\text{Ext}_\Lambda^*(M, \Lambda/\text{rad } \Lambda)$ is finitely generated.*
- (iii) *For all $N \in \text{mod } \Lambda$, the H -module $\text{Ext}_\Lambda^*(N, M)$ is finitely generated.*
- (iv) *The H -module $\text{Ext}_\Lambda^*(\Lambda/\text{rad } \Lambda, M)$ is finitely generated.*

Then the implications (i) \Leftrightarrow (ii) and (iii) \Leftrightarrow (iv) hold.

Proof. We prove only the implication (ii) \Rightarrow (i); the implication (iv) \Rightarrow (iii) is proved analogously. The proof is by induction on the length $\ell(N)$ of a module N . Since the H -module $\text{Ext}_\Lambda^*(M, \Lambda/\text{rad } \Lambda)$ is finitely generated, so is $\text{Ext}_\Lambda^*(M, S)$ for any simple Λ -module S . Now suppose $\ell(N) > 1$, and choose a nonzero proper submodule L of N . The exact sequence

$$0 \rightarrow L \rightarrow N \rightarrow N/L \rightarrow 0$$

induces an exact sequence

$$\text{Ext}_\Lambda^*(M, L) \rightarrow \text{Ext}_\Lambda^*(M, N) \rightarrow \text{Ext}_\Lambda^*(M, N/L)$$

of H -modules. By assumption, both the end terms are finitely generated H -modules, hence so is the middle term since H is Noetherian. \square

There are situations when finiteness always occurs, regardless of the module M we start with. Namely, when all the cohomology groups of the algebra are finitely generated over a central ring of cohomology operators, as in the following definition.

Definition. The algebra Λ satisfies **Fg** if there exists a commutative Noetherian graded k -algebra $R = \bigoplus_{i=0}^{\infty} R_i$ of finite type (i.e. $\dim_k R_i < \infty$ for all i) satisfying the following:

- (i) For every $X \in \text{mod } \Lambda$ there is a graded ring homomorphism

$$\phi_X: R \rightarrow \text{Ext}_\Lambda^*(X, X).$$

- (ii) For each pair (X, Y) of finitely generated Λ -modules, the scalar actions from R on $\text{Ext}_\Lambda^*(X, Y)$ via ϕ_X and ϕ_Y coincide, and $\text{Ext}_\Lambda^*(X, Y)$ is a finitely generated R -module.

As mentioned, this holds if Λ is the group algebra of a finite group, a cocommutative Hopf algebra, a finite dimensional commutative complete intersection, or if the Hochschild cohomology ring of Λ is suitably “nice” (cf. [2], [3], [4], [7], [12], [14], [10], [11], [6]). Now suppose Λ satisfies **Fg** with respect to a graded ring R as in the definition, and let X be a Λ -module. Then $\phi_X(R)$ is a commutative Noetherian graded subalgebra of the center of $\text{Ext}_\Lambda^*(X, X)$. Moreover, for any $Y \in \text{mod } \Lambda$ both $\text{Ext}_\Lambda^*(X, Y)$ and $\text{Ext}_\Lambda^*(Y, X)$ are finitely generated $\phi_X(R)$ -modules.

When Λ satisfies **Fg**, we may also define support varieties with respect to the ring of cohomology operators. Namely, let R be as in the definition. Given Λ -modules X and Y , we define

$$V_R(X, Y) \stackrel{\text{def}}{=} \{\mathfrak{m} \in \text{MaxSpec } R \mid \text{Ann}_R \text{Ext}_\Lambda^*(X, Y) \subseteq \mathfrak{m}\}.$$

Is this variety comparable to $V_{\phi_X(R)}^i(Y)$ and $V_{\phi_Y(R)}^p(X)$, i.e. the varieties defined by the ideals $\text{Ann}_{\phi_X(R)}^i(Y) = \text{Ann}_{\phi_X(R)} \text{Ext}_{\Lambda}^*(X, Y)$ and $\text{Ann}_{\phi_Y(R)}^p(X) = \text{Ann}_{\phi_Y(R)} \text{Ext}_{\Lambda}^*(X, Y)$, respectively? The following result shows that the three varieties $V_{\phi_X(R)}^i(Y)$, $V_{\phi_Y(R)}^p(X)$ and $V_R(X, Y)$ are in fact isomorphic.

Proposition 3.4. *Suppose Λ satisfies **Fg** with respect to a graded ring R as in the definition above, and let X and Y be Λ -modules. Then the varieties $V_{\phi_X(R)}^i(Y)$, $V_{\phi_Y(R)}^p(X)$ and $V_R(X, Y)$ are isomorphic.*

Proof. Let \mathfrak{m} be a maximal ideal in R . Since $\phi_X(\text{Ann}_R \text{Ext}_{\Lambda}^*(X, Y))$ equals $\text{Ann}_{\phi_X(R)}^i Y$, we see that $\text{Ann}_R \text{Ext}_{\Lambda}^*(X, Y) \subseteq \mathfrak{m}$ if and only if $\text{Ann}_{\phi_X(R)}^i Y \subseteq \phi_X(\mathfrak{m})$. Therefore \mathfrak{m} belongs to $V_R(X, Y)$ if and only if $\phi_X(\mathfrak{m})$ belongs to $V_{\phi_X(R)}^i(Y)$, and this shows that the varieties $V_R(X, Y)$ and $V_{\phi_X(R)}^i(Y)$ are isomorphic. Similarly the varieties $V_R(X, Y)$ and $V_{\phi_Y(R)}^p(X)$ are isomorphic. \square

As we saw above, when Λ satisfies **Fg** then for every Λ -module M there exists a commutative Noetherian graded subalgebra $H \subseteq \text{Ext}_{\Lambda}^*(M, M)$ over which $\text{Ext}_{\Lambda}^*(\Lambda/\text{rad } \Lambda, M)$ and $\text{Ext}_{\Lambda}^*(M, \Lambda/\text{rad } \Lambda)$ are finitely generated. However, the following example shows that this may very well hold for a module even if the algebra does not satisfy **Fg**.

Example. Suppose Λ is selfinjective, and let M be a nonzero periodic Λ -module, i.e. $\Omega_{\Lambda}^p(M) \simeq M$ for some $p \geq 1$. Then the first part of the minimal projective resolution of M is a p -fold extension

$$0 \rightarrow M \rightarrow P_{p-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Denote this extension by μ , and consider the subalgebra $k[\mu]$ of $\text{Ext}_{\Lambda}^*(M, M)$. This subalgebra is a Noetherian ring over which $\text{Ext}_{\Lambda}^*(M, M)$ is finitely generated as a module. In fact, given any Λ -module N , the $k[\mu]$ -modules $\text{Ext}_{\Lambda}^*(M, N)$ and $\text{Ext}_{\Lambda}^*(N, M)$ are finitely generated (cf. [15] and [16] for a discussion of these phenomena). As an example, consider the quantum exterior algebra

$$k\langle x, y \rangle / (x^2, xy - qyx, y^2),$$

where the element q is a nonzero non-root of unity in k . Let M be a two dimensional vector space with basis $\{u, v\}$, say. By defining

$$xu = 0, \quad xv = 0, \quad yu = v, \quad yv = 0,$$

this vector space becomes a module over the quantum exterior algebra. Moreover, it is not difficult to see that this module is periodic of period one (cf. [5, Example 4.5]). However, by [11] and [6, Theorem 5.5] the algebra does not satisfy **Fg**, since q is not a root of unity.

We now return to the general theory. A natural question to ask is how big the relative support variety of a module is. For an arbitrary module N , this cannot be answered unless we introduce finiteness conditions, since a priori there is no relationship between H and $\text{Ext}_{\Lambda}^*(M, N)$ or $\text{Ext}_{\Lambda}^*(N, M)$. However, when we introduce finite generation, the situation becomes much more manageable.

Let X be a Λ -module with minimal projective and injective resolutions

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0,$$

$$0 \rightarrow X \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots,$$

say. Then we define the *complexity* and *plexity* of X , denoted $\text{cx } X$ and $\text{px } X$, respectively, as

$$\begin{aligned}\text{cx } X &\stackrel{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \dim_k P_n \leq an^{t-1} \text{ for } n \gg 0\}, \\ \text{px } X &\stackrel{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \dim_k I^n \leq an^{t-1} \text{ for } n \gg 0\}.\end{aligned}$$

The complexity and the plexity of a module are not necessarily finite. Also, from the definition we see that $\text{cx } X = 0$ (respectively, $\text{px } X = 0$) if and only if X has finite projective dimension (respectively, finite injective dimension). It is well known that the complexity of X equals $\gamma(\text{Ext}_\Lambda^*(X, \Lambda/\text{rad } \Lambda))$, whereas its plexity equals $\gamma(\text{Ext}_\Lambda^*(\Lambda/\text{rad } \Lambda, X))$. Generalizing this, we define the *complexity of the pair* (X, Y) of Λ -modules to be $\gamma(\text{Ext}_\Lambda^*(X, Y))$, and denote it by $\text{cx}(X, Y)$. Thus $\text{cx } X$ is the complexity of the pair $(X, \Lambda/\text{rad } \Lambda)$, whereas $\text{px } X$ is the complexity of the pair $(\Lambda/\text{rad } \Lambda, X)$. Note that $\text{cx}(X, Y) \neq \text{cx}(Y, X)$ in general, that is, the order matters.

It follows from the discussion prior to [4, Proposition 5.3.5] that $\text{cx}(M, N) \leq \text{cx } M$, and similarly $\text{cx}(M, N) \leq \text{px } N$. In particular $\text{cx}(M, M)$ is at most $\text{cx } M$ and $\text{px } M$. The following example, using the same quantum exterior algebra as in the above example, shows that in general the inequalities might be strict.

Example. Let q be a nonzero non-root of unity in k , and let Γ be the quantum exterior algebra

$$k\langle x, y \rangle / (x^2, xy - qyx, y^2).$$

Denote by X the Γ -module $\Gamma(x + y)$. It was shown in [17] that the complex

$$\dots \rightarrow \Gamma \xrightarrow{\cdot(x+q^2y)} \Gamma \xrightarrow{\cdot(x-xy)} \Gamma \xrightarrow{\cdot(x+y)} X \rightarrow 0$$

is a minimal projective resolution of X , and that $\text{Ext}_\Gamma^n(X, X) = 0$ for $n \gg 0$. Thus $\text{cx } X = 1$, whereas $\text{cx}(X, X) = 0$.

The following result shows that equality occurs when finite generation holds. Recall that H is a commutative Noetherian graded subalgebra of $\text{Ext}_\Lambda^*(M, M)$.

Proposition 3.5. [4, Proposition 5.3.5] *If the H -module $\text{Ext}_\Lambda^*(M, \Lambda/\text{rad } \Lambda)$ is finitely generated, then*

$$\text{cx } M = \text{cx}(M, M) = \gamma(H).$$

Similarly, if the H -module $\text{Ext}_\Lambda^(\Lambda/\text{rad } \Lambda, M)$ is finitely generated, then*

$$\text{px } M = \text{cx}(M, M) = \gamma(H).$$

In particular, if both $\text{Ext}_\Lambda^(M, \Lambda/\text{rad } \Lambda)$ and $\text{Ext}_\Lambda^*(\Lambda/\text{rad } \Lambda, M)$ are finitely generated over H , then $\text{cx } M = \text{px } M$.*

As for the “size” of the relative support varieties, the following result shows that it is given in terms of the complexity, provided finite generation holds.

Proposition 3.6. *If the H -module $\text{Ext}_\Lambda^*(M, N)$ is finitely generated, then $\dim V_H^i(N) = \text{cx}(M, N)$. Similarly, if the H -module $\text{Ext}_\Lambda^*(N, M)$ is finitely generated, then $\dim V_H^p(N) = \text{cx}(N, M)$.*

Proof. If $\text{Ext}_\Lambda^*(M, N)$ is finitely generated over H , then $\gamma(H/\text{Ann}_H^i N) = \gamma(\text{Ext}_\Lambda^*(M, N))$, and so by definition $\dim V_H^i(N) = \text{cx}(M, N)$. The other equality is proved similarly. \square

4. WILD ALGEBRAS AND COMPLEXITY

In this section we assume that our field k is algebraically closed. Recall that Λ is of *finite representation type* if there are only finitely many non-isomorphic indecomposable Λ -modules. Furthermore, recall that Λ is of *tame representation type* if there exist infinitely many non-isomorphic indecomposable Λ -modules, but they all belong to one-parameter families, and in each dimension there are finitely many such families. Finally, the algebra Λ is of *wild representation type* if it is not of finite or tame type.

In [9], Crawley-Boevey established a link between the representation type of a selfinjective finite dimensional algebra and the complexities of its modules. Namely, it was shown that for such an algebra, in any dimension only finitely many indecomposable modules are not of complexity one. Using this, Farnsteiner showed in [13] that the complexity of every module of a tame block of a finite group scheme is at most two.

Suppose our algebra Λ is selfinjective and satisfies the “global” finite generation hypothesis **Fg** defined immediately after Proposition 3.3. That is, suppose there exists a commutative Noetherian graded k -algebra $R = \bigoplus_{i=0}^{\infty} R_i$ of finite type satisfying the following:

- (i) For every $X \in \text{mod } \Lambda$ there is a graded ring homomorphism

$$\phi_X: R \rightarrow \text{Ext}_{\Lambda}^*(X, X).$$

- (ii) For each pair (X, Y) of finitely generated Λ -modules, the scalar actions from R on $\text{Ext}_{\Lambda}^*(X, Y)$ via ϕ_X and ϕ_Y coincide, and $\text{Ext}_{\Lambda}^*(X, Y)$ is a finitely generated R -module.

Then Farnsteiner’s proof still applies; hence Λ is wild if there exists a module of complexity at least three. We end this paper with the following result, which generalizes this.

Theorem 4.1. *Suppose Λ is selfinjective, and there exists a Λ -module M satisfying the following:*

- (i) $\text{cx}(M, M) \geq 3$,
(ii) *there exists a commutative Noetherian graded subalgebra $H \subseteq \text{Ext}_{\Lambda}^*(M, M)$ over which $\text{Ext}_{\Lambda}^*(M, M)$ is a finitely generated module.*

Then Λ is of wild representation type.

Proof. Suppose (i) holds. By Proposition 3.2, we may assume that H is a polynomial ring, say $H = k[y_1, \dots, y_n]$, where $n \geq 3$. Denote the ideal $\text{Ann}_H^i N \subseteq H$ by \mathfrak{a} . Since $\text{Ext}_{\Lambda}^*(M, M)$ is a finitely generated H/\mathfrak{a} -module, we may apply the Noether normalization lemma and obtain a new polynomial ring $R = k[x_1, \dots, x_c] \subseteq H/\mathfrak{a}$, with $c = \text{cx}(M, M)$, over which $\text{Ext}_{\Lambda}^*(M, M)$ is finitely generated. Moreover, we may assume that the homogeneous elements x_1, \dots, x_c are of the same degree, say $|x_i| = d$.

The maximal ideals in R correspond to points $(\alpha_1, \dots, \alpha_c) \in k^c$. Given an ideal $I \subseteq R$, we denote its variety by $V_R(I)$, thus

$$V_R(I) = \{\alpha \in k^c \mid f(\alpha) = 0 \text{ for all } f \in I\}.$$

Now for each $\alpha \in k$, denote the element $x_1 + \alpha x_2 \in R$ by x_{α} . Lifting this element to H gives a homogeneous element $\eta_{\alpha} \in \text{Ext}_{\Lambda}^*(M, M)$ of degree d , from which we obtain a short exact sequence

$$(\dagger) \quad 0 \rightarrow M \rightarrow K_{\alpha} \rightarrow \Omega_{\Lambda}^{d-1}(M) \rightarrow 0.$$

By applying the same proof as in [10, Proposition 4.3], we see that $V_R^p(K_\alpha) = V_R(x_1 + \alpha x_2)$. Thus $\text{cx}(K_\alpha, M) = c - 1$, and K_α is not isomorphic to $K_{\alpha'}$ whenever $\alpha \neq \alpha'$.

For each α , let $K_\alpha = K_\alpha^1 \oplus \cdots \oplus K_\alpha^{t_\alpha}$ be a decomposition of K_α into indecomposable Λ -modules. Moreover, denote the ideal $\text{Ann}_R^p K_\alpha^i \subseteq R$, that is, the ideal $\text{Ann}_R \text{Ext}_\Lambda^*(K_\alpha^i, M)$, by \mathfrak{a}_i . Then $V_R^p(K_\alpha^i)$ is by definition the variety $V_R(\mathfrak{a}_i)$, and therefore

$$V_R(x_1 + \alpha x_2) = V_R^p(K_\alpha) = \bigcup_{i=1}^{t_\alpha} V_R^p(K_\alpha^i) = \bigcup_{i=1}^{t_\alpha} V_R(\mathfrak{a}_i) = V_R\left(\prod_{i=1}^{t_\alpha} \mathfrak{a}_i\right),$$

which in turn implies

$$\prod_{i=1}^{t_\alpha} \mathfrak{a}_i \subseteq \sqrt{\prod_{i=1}^{t_\alpha} \mathfrak{a}_i} = \sqrt{(x_1 + \alpha x_2)}.$$

Since the ideal $(x_1 + \alpha x_2)$ is prime, it is equal to its own radical, and contains one of the ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_{t_\alpha}$, say \mathfrak{a}_1 . However, the variety $V_R^p(K_\alpha^1)$ is contained in $V_R^p(K_\alpha)$, and therefore $(x_1 + \alpha x_2) = \sqrt{\mathfrak{a}_1}$. Consequently $\sqrt{\mathfrak{a}_1} = (x_1 + \alpha x_2)$, and this shows that the varieties $V_R^p(K_\alpha^1)$ and $V_R^p(K_\alpha)$ are equal.

The indecomposable Λ -modules $\{K_\alpha^1\}_{\alpha \in k}$ are pairwise nonisomorphic, and from the exact sequence (†) we see that $\dim_k K_\alpha^1 \leq \dim_k M + \dim_k \Omega_\Lambda^{d-1}(M)$ for every $\alpha \in k$. Moreover, by construction we know that $\text{cx}(K_\alpha^1, M) = \text{cx}(K_\alpha, M)$, hence

$$2 \leq c - 1 = \text{cx}(K_\alpha^1, M) \leq \text{cx} K_\alpha^1.$$

The result now follows from Crawley-Boevey's result [9, Theorem D]. \square

Using Proposition 3.3 and Proposition 3.5, we obtain the following corollaries to Theorem 4.1.

Corollary 4.2. *Suppose Λ is selfinjective, and there exists a Λ -module M satisfying the following:*

- (i) $\text{cx} M \geq 3$,
- (ii) *there exists a commutative Noetherian graded subalgebra $H \subseteq \text{Ext}_\Lambda^*(M, M)$ over which $\text{Ext}_\Lambda^*(M, \Lambda/\text{rad } \Lambda)$ is a finitely generated module.*

Then Λ is of wild representation type.

Corollary 4.3. *Suppose Λ is selfinjective, and there exists a Λ -module M satisfying the following:*

- (i) $\text{px} M \geq 3$,
- (ii) *there exists a commutative Noetherian graded subalgebra $H \subseteq \text{Ext}_\Lambda^*(M, M)$ over which $\text{Ext}_\Lambda^*(\Lambda/\text{rad } \Lambda, M)$ is a finitely generated module.*

Then Λ is of wild representation type.

Finally, by applying Proposition 3.1 we obtain the following corollary to Theorem 4.1. It shows that an algebra is wild if it possesses a module whose Ext-algebra is “big enough”, Noetherian and finitely generated over its center.

Corollary 4.4. *Suppose Λ is selfinjective, and there exists a Λ -module M satisfying the following:*

- (i) $\text{cx}(M, M) \geq 3$,
- (ii) *$\text{Ext}_\Lambda^*(M, M)$ is a Noetherian ring and finitely generated as a module over its center.*

Then Λ is of wild representation type.

We end this paper with the following example illustrating Theorem 4.1, an example in which the algebra does not satisfy the finite generation hypothesis **Fg**. Namely, we use the same quantum exterior algebra as in the examples in Section 3

Example. Let q be a nonzero non-root of unity in k , and denote by Γ the quantum exterior algebra

$$k\langle x, y \rangle / (x^2, xy - qyx, y^2).$$

Let X be the Γ -module from the example following Proposition 3.4, i.e. X is a two dimensional vector space with basis $\{u, v\}$, say, and with scalar multiplication defined by

$$xu = 0, \quad xv = 0, \quad yu = v, \quad yv = 0.$$

This module is periodic of period one, and so if we denote its projective cover

$$0 \rightarrow X \rightarrow P \rightarrow X \rightarrow 0$$

by μ , the Ext-algebra $\text{Ext}_\Gamma^*(X, X)$ is finitely generated as a module over the polynomial subalgebra $k[\mu]$. Now let Λ be the algebra $\Gamma \otimes_k \Gamma \otimes_k \Gamma$, and let M be the Λ -module $X \otimes_k X \otimes_k X$. Then the Ext-algebra of M is given by

$$\text{Ext}_\Lambda^*(M, M) = \text{Ext}_\Gamma^*(X, X) \overline{\otimes}_k \text{Ext}_\Gamma^*(X, X) \overline{\otimes}_k \text{Ext}_\Gamma^*(X, X),$$

where $\overline{\otimes}$ differs from the usual tensor product only in that elements of odd degree anticommute (cf. [8, Chapter XI]). Therefore $\text{Ext}_\Lambda^*(M, M)$ is finitely generated as a module over a commutative Noetherian graded subalgebra. Moreover, since $\text{cx}_\Gamma(X, X) = 1$, we see that $\text{cx}_\Lambda(M, M) = 3$. Finally, the algebra Λ does not satisfy **Fg**. Namely, this algebra is a quantum exterior algebra on six generators, where some of the defining commutators equal q . Then by [11] and [6, Theorem 5.5] Λ does not satisfy **Fg**, since q is not a root of unity.

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