REPRESENTATION DIMENSION AND FINITELY GENERATED COHOMOLOGY

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Dedicated to Karin Erdmann on the occasion of her sixtieth birthday

Abstract. We consider selfinjective Artin algebras whose cohomology groups are finitely generated over a central ring of cohomology operators. For such an algebra, we show that the representation dimension is strictly greater than the maximal complexity occurring among its modules. This provides a unified approach to computing lower bounds for the representation dimension of group algebras, exterior algebras and Artin complete intersections. We also obtain new examples of classes of algebras with arbitrarily large representation dimension.

1. Introduction

In his 1971 notes [Au1], Auslander introduced the notion of the representation dimension of an Artin algebra. This invariant measures how far an algebra is from having finite representation type; it was introduced in order to study algebras of infinite representation type. A non-semisimple algebra is of finite type if and only if its representation dimension is exactly two, and of infinite type if and only if the representation dimension is at least three.

The representation dimension of an Artin algebra is always finite (cf. [Iya]). However, for a long time it was unclear whether there could exist algebras of representation dimension strictly greater than three. Moreover, Igusa and Todorov showed in [IgT] that if this was not the case, i.e. if the representation dimension could not exceed three, then the finitistic dimension conjecture would hold. However, in 2006 Rouquier showed in [Ro2] that the representation dimension of the exterior algebra on a d-dimensional vector space is $d+1$, using the notion of the dimension of a triangulated category (cf. [Ro1]). Other examples illustrating this were subsequently given in [AvI], [BeO], [KrK], [Op1] and [Op2].

In this paper, we study selfinjective Artin algebras satisfying a certain finite generation hypothesis on its cohomology groups. Namely, we assume there exists a commutative graded Noetherian ring over which all the cohomology groups are finitely generated. Using a version of the Ghost Lemma, we obtain a lower bound for the dimension of the stable module category of an algebra satisfying this hypothesis. In particular, we show that the representation dimension of such an algebra is strictly greater than the maximal complexity occurring among its modules. This provides a unified approach to computing the known lower bounds for the representation dimension of group algebras, exterior algebras and Artin complete intersections. We also obtain new examples of classes of algebras with arbitrarily large representation dimension.

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2. Representation dimension

Throughout this paper, we let k be a commutative Artin ring and Λ an Artin k-algebra with Jacobson radical $\mathfrak{r}$. We denote by $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules. The *representation dimension* of Λ, denoted $\operatorname{repdim}\Lambda$, is defined as

$$\operatorname{repdim}\Lambda \stackrel{\text{def}}{=} \inf\{\operatorname{gl.dim}\operatorname{End}_\Lambda(M) \mid M \text{ generates and cogenerates } \operatorname{mod}\Lambda\},$$

where gl. dim denotes the global dimension of an algebra. Auslander showed that the representation dimension of a selfinjective algebra is at most its Loewy length, whereas Iyama showed in [Iya] that this invariant is finite for every Artin algebra.

In order to compute the representation dimension of exterior algebras, Rouquier used the notion of the dimension of a triangulated category, a concept he introduced in [Ro1]. We recall here the definitions. Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of $\mathcal{T}$. We denote by $\langle \mathcal{C} \rangle$ the full subcategory of $\mathcal{T}$ consisting of all the direct summands of finite direct sums of shifts of objects in $\mathcal{C}$. Furthermore, we denote by $\mathcal{C} * \mathcal{D}$ the full subcategory of $\mathcal{T}$ consisting of objects M such that there exists a distinguished triangle

$$C \to M \to D \to C[1]$$

in $\mathcal{T}$, with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Finally, we denote the subcategory $\langle \mathcal{C} * \mathcal{D} \rangle$ by $\mathcal{C} \diamond \mathcal{D}$. Now define $\langle \mathcal{C} \rangle_1$ to be $\langle \mathcal{C} \rangle$, and for each $n \geq 2$ define inductively $\langle \mathcal{C} \rangle_n$ to be $\langle \mathcal{C} \rangle_{n-1} \diamond \langle \mathcal{C} \rangle$. The *dimension* of $\mathcal{T}$, denoted $\dim \mathcal{T}$, is defined as

$$\dim \mathcal{T} \stackrel{\text{def}}{=} \inf\{d \in \mathbb{Z} \mid \text{ there exists an object } M \in \mathcal{T} \text{ such that } \mathcal{T} = \langle M \rangle_{d+1}\}.$$

In other words, the dimension of $\mathcal{T}$ is the minimum number of layers needed to obtain $\mathcal{T}$ from one of its objects.

The key ingredient in the proof of our main result is the following lemma on compositions of natural transformations. The lemma is analogous to [Ro1, Lemma 4.11].

Lemma 2.1 (Ghost Lemma). *Let $\mathcal{T}$ be a triangulated category, let $H_1, \dots, H_{n+1}$ be cohomological functors on $\mathcal{T}$, and for each $1 \leq i \leq n$ let $H_i \xrightarrow{f_i} H_{i+1}$ be a natural transformation. Furthermore, let $\mathcal{C}_1, \dots, \mathcal{C}_n$ be subcategories of $\mathcal{T}$ closed under shifts, and assume that for every object $c \in \mathcal{C}_i$ the map $H_i(c[j]) \xrightarrow{f_i} H_{i+1}(c[j])$ vanishes for $j \gg 0$ (respectively, for $j \ll 0$). Then for every object $w \in \mathcal{C}_1 \diamond \cdots \diamond \mathcal{C}_n$ the map $H_1(w[j]) \xrightarrow{f_n \cdots f_1} H_{n+1}(w[j])$ vanishes for $j \gg 0$ (respectively, for $j \ll 0$).*

Proof. We may assume $n \geq 2$. Let $c_1 \to c \to c_2 \to c_1[1]$ be a triangle in $\mathcal{T}$ with $c_1 \in \mathcal{C}_1$ and $c_2 \in \mathcal{C}_2$. Then for every $j \in \mathbb{Z}$, there is a commutative diagram

$$\begin{array}{ccccc} H_1(c_1[j]) & \longrightarrow & H_1(c[j]) & \longrightarrow & H_1(c_2[j]) \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} \\ H_2(c_1[j]) & \longrightarrow & H_2(c[j]) & \longrightarrow & H_2(c_2[j]) \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_2} \\ H_3(c_1[j]) & \longrightarrow & H_3(c[j]) & \longrightarrow & H_3(c_2[j]) \end{array}$$

with exact rows. By assumption, there is an integer j_1 such that the vertical upper left map vanishes for $j \geq j_1$, and an integer j_2 such that the vertical lower right map vanishes for $j \geq j_2$. An easy diagram chase shows that the vertical middle composition vanishes for $j \geq \max\{j_1, j_2\}$, hence for every object $w \in \mathcal{C}_1 \diamond \mathcal{C}_2$ the map $H_1(w[j]) \xrightarrow{f_2 f_1} H_3(w[j])$ vanishes for $j \gg 0$. An induction argument now establishes the lemma. □

The triangulated category we shall use is the *stable module category* of Λ, in the case when Λ is selfinjective. This category, denoted $\underline{\operatorname{mod}}\Lambda$, is defined as follows: the objects of $\underline{\operatorname{mod}}\Lambda$ are the same as in $\operatorname{mod}\Lambda$, but two morphisms in $\operatorname{mod}\Lambda$ are equal in $\underline{\operatorname{mod}}\Lambda$ if their difference factors through a projective Λ-module. The cosyzygy functor $\Omega_\Lambda^{-1}\colon \underline{\operatorname{mod}}\Lambda \to \underline{\operatorname{mod}}\Lambda$ is an equivalence of categories, and a triangulation of $\underline{\operatorname{mod}}\Lambda$ is given by using this functor as a shift and by letting short exact sequences in $\operatorname{mod}\Lambda$ correspond to triangles. Thus $\underline{\operatorname{mod}}\Lambda$ is a triangulated category, and its dimension is related to the representation dimension of Λ by the following result.

Proposition 2.2. [Ro2, Proposition 3.7] *If Λ is selfinjective and not semisimple, then* $\operatorname{repdim}\Lambda \geq \dim(\underline{\operatorname{mod}}\Lambda) + 2$.

This result was originally formulated for a finite dimensional algebra over a field, but it works just as well in our setting, i.e. for an Artin algebra which is not necessarily finite dimensional over a field.

The main result of this paper relates the representation dimension of Λ with the maximal complexity occurring among its finitely generated modules. Recall therefore that for a module $M \in \operatorname{mod}\Lambda$ with minimal projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

say, the *complexity* of M is defined as

$$\operatorname{cx} M \stackrel{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \ell_k(P_n) \leq a n^{t-1} \text{ for } n \gg 0\}.$$

In general, the complexity of a module may be infinite, whereas it is zero if and only if the module is projective. The complexity of M can be computed as the rate of growth of the graded k-module $\operatorname{Ext}_\Lambda^*(M, \Lambda/\mathfrak{r})$, and from the definition we also see that it equals the complexity of $\Omega_\Lambda^i(M)$ for any $i \in \mathbb{N}$. Moreover, given a short exact sequence

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

in $\operatorname{mod}\Lambda$, it is well known that the inequality $\operatorname{cx} X_u \leq \sup\{\operatorname{cx} X_v, \operatorname{cx} X_w\}$ holds for $\{u, v, w\} = \{1, 2, 3\}$. In particular, induction on the length of a module shows that $\operatorname{cx} X \leq \operatorname{cx} \Lambda/\mathfrak{r}$ for every $X \in \operatorname{mod}\Lambda$. We end this section with the following elementary lemma, which shows that a module generating $\underline{\operatorname{mod}}\Lambda$ must be of maximal complexity.

Lemma 2.3. *Let Λ be selfinjective, let $M \in \operatorname{mod}\Lambda$ be a module, and suppose there exists a number $n \in \mathbb{N}$ such that $\langle M \rangle_n = \underline{\operatorname{mod}}\Lambda$. Then $\operatorname{cx} N \leq \operatorname{cx} M$ for every $N \in \operatorname{mod}\Lambda$, in particular $\operatorname{cx} M = \operatorname{cx}\Lambda/\mathfrak{r}$.*

Proof. The result follows from the fact that triangles in $\underline{\operatorname{mod}}\Lambda$ correspond to short exact sequences in $\operatorname{mod}\Lambda$. □

3. Finitely generated cohomology

We now introduce a certain "finite generation" assumption on the cohomology groups of Λ. Recall that for Λ-modules X and Y, the graded k-module $\operatorname{Ext}_\Lambda^*(X, Y)$ is an $\operatorname{Ext}_\Lambda^*(Y, Y) - \operatorname{Ext}_\Lambda^*(X, X)$-bimodule via Yoneda products. Also recall that a graded k-module $\bigoplus V_i$ is of *finite type* if each V_i is a finitely generated k-module.

Assumption (Fg). There exists a commutative Noetherian graded k-algebra $H = \bigoplus_{i=0}^{\infty} H^i$ of finite type satisfying the following:

(i) For every $M \in \operatorname{mod}\Lambda$ there is a graded ring homomorphism

$$\phi_M \colon H \to \operatorname{Ext}_\Lambda^*(M, M).$$

(ii) For each pair (X,Y) of finitely generated Λ-modules, the scalar actions from H on $\operatorname{Ext}^*_\Lambda(X,Y)$ via ϕ_X and ϕ_Y coincide, and $\operatorname{Ext}^*_\Lambda(X,Y)$ is a finitely generated H-module.

In the assumption, why do we require that the left and right scalar multiplications on $\operatorname{Ext}^*_\Lambda(X,Y)$ coincide? The reason is that this requirement is what makes the bifunctor $\operatorname{Ext}^*_\Lambda(-,-)$ preserve maps. To see this, let $f\colon M \to M'$ be a homomorphism in $\operatorname{mod}\Lambda$. For every $N \in \operatorname{mod}\Lambda$, this map induces a homomorphism $\hat{f}\colon \operatorname{Ext}^*_\Lambda(M',N) \to \operatorname{Ext}^*_\Lambda(M,N)$ of graded groups. The image of a homogeneous element

$$\theta\colon 0 \to N \to X_n \to \cdots \to X_1 \to M' \to 0$$

is the extension θf given by the commutative diagram

$$\begin{array}{ccccccccccccc}
\theta f\colon 0 & \longrightarrow & N & \longrightarrow & X_n & \longrightarrow \cdots \longrightarrow & X_2 & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \downarrow & & \downarrow f & & \\
\theta\colon 0 & \longrightarrow & N & \longrightarrow & X_n & \longrightarrow \cdots \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & M' & \longrightarrow & 0
\end{array}$$

in which the module Y is a pullback. For a homogeneous element $\eta \in H$ we then get

$$\begin{aligned}
\hat{f}(\theta \cdot \eta) &= \hat{f}(\eta \cdot \theta) \\
&= \hat{f}\left(\phi_N(\eta) \circ \theta\right) \\
&= \phi_N(\eta) \circ (\theta f) \\
&= \eta \cdot \hat{f}(\theta) \\
&= \hat{f}(\theta) \cdot \eta,
\end{aligned}$$

showing $\hat{f}$ is a homomorphism of H-modules. Similarly, $\operatorname{Ext}^*_\Lambda(-,-)$ preserves maps in the second argument. The fact that $\operatorname{Ext}^*_\Lambda(-,-)$ preserves maps is absolutely essential. Note that when using this property, induction on length of modules shows that the finite generation part of the assumption **Fg** is equivalent to $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ being a finitely generated H-module.

It should also be noted that when **Fg** holds, then every finitely generated Λ-module has finite complexity, i.e. $\operatorname{cx}\Lambda/\mathfrak{r} < \infty$. Namely, the H-module $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ is finitely generated, and so its rate of growth is not more than that of H. The ring H is commutative Noetherian and of finite type, and hence its rate of growth is finite.

In the following examples we point out three important situations in which the assumption **Fg** holds.

Examples. (i) Suppose k is a field of positive characteristic p, and let G be a finite group whose order is divisible by p. Then by a theorem of Evens (cf. [Eve]), the graded commutative group cohomology ring $\operatorname{H}^*(G,k) = \operatorname{Ext}^*_{kG}(k,k)$ is Noetherian. Moreover, if X_1 and X_2 are finitely generated kG-modules, then $\operatorname{Ext}^*_{kG}(X_1,X_2)$ is a finitely generated $\operatorname{H}^*(G,k)$-module via the maps

$$- \otimes_k X_i\colon \operatorname{H}^*(G,k) \to \operatorname{Ext}^*_{kG}(X_i,X_i),$$

and the right and left scalar actions induced by these maps commute up to a graded sign. The even part $\bigoplus H^{2i}(G,k)$ of $\operatorname{H}^*(G,k)$ is a commutative k-algebra, over which $\operatorname{H}^*(G,k)$ is finitely generated as a module.

(ii) Let $(A,\mathfrak{m},k)$ be a commutative Noetherian local complete intersection. That is, the completion $\widehat{A}$, with respect to the maximal ideal $\mathfrak{m}$, is of the form $R/(x_1,\dots,x_c)$, where R is regular local and $x_1,\dots,x_c$ is a regular sequence. We may without loss of generality assume that the length c of the defining regular sequence is the codimension of A, i.e. $c = \dim_k \left(\mathfrak{m} / \mathfrak{m}^2\right) - \dim A$. By [Avr, Section 1] there exists a polynomial ring $\widehat{A}[\chi_1, \dots, \chi_c]$ in commuting *Eisenbud operators*, such that for every finitely generated $\widehat{A}$-module X there is a homomorphism

$$\phi_X \colon \widehat{A}[\chi_1, \dots, \chi_c] \to \operatorname{Ext}^*_{\widehat{A}}(X, X)$$

of graded rings. Moreover, for every finitely generated $\widehat{A}$-module Y, the left and right scalar actions on $\operatorname{Ext}^*_{\widehat{A}}(X, Y)$ coincide, and the latter is a finitely generated $\widehat{A}[\chi_1, \dots, \chi_c]$-module. Now if A is Artin, then it is a complete ring since $\mathfrak{m}$ is nilpotent. Thus **Fg** holds in this case.

(iii) Suppose Λ is projective as a k-module. Denote by Λ^{e} the enveloping algebra $\Lambda \otimes_k \Lambda^{\mathrm{op}}$ of Λ, and by $\operatorname{HH}^*(\Lambda)$ its Hochschild cohomology ring. By [Yon, Proposition 3], this is a graded commutative ring, and equal to $\operatorname{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda)$ since Λ is k-projective. If X_1 and X_2 are finitely generated Λ-modules, then the right and left scalar actions from $\operatorname{HH}^*(\Lambda)$ on $\operatorname{Ext}^*_\Lambda(X_1, X_2)$, via the maps

$$- \otimes_\Lambda X_i \colon \operatorname{HH}^*(\Lambda) \to \operatorname{Ext}^*_\Lambda(X_i, X_i),$$

are graded commutative.

In [EHSST] the finite generation assumption imposed was the following: there exists a commutative Noetherian graded subalgebra $S = \bigoplus_{i=0}^\infty S^i$ of $\operatorname{HH}^*(\Lambda)$, with $S^0 = \operatorname{HH}^0(\Lambda)$, such that $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated S-module. However, having to deal with such an "unknown" subalgebra of the Hochschild cohomology ring is not satisfactory, and in fact it is not difficult to see (cf. [Sol, Proposition 5.7]) that the assumption is equivalent to the following one: the Hochschild cohomology ring $\operatorname{HH}^*(\Lambda)$ is Noetherian and $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated $\operatorname{HH}^*(\Lambda)$-module. Therefore, as in the first example, we see that **Fg** holds by choosing H to be the even part $\bigoplus \operatorname{HH}^{2i}(\Lambda)$ of $\operatorname{HH}^*(\Lambda)$.

In particular, the assumption **Fg** holds for exterior algebras. Namely, suppose k is a field, let n be a number, and denote by Λ the algebra

$$k\langle x_1, \dots, x_n \rangle / (x_i^2, x_i x_j + x_j x_i),$$

i.e. Λ is the exterior algebra on an n-dimensional vector space. Then by [Sol, Theorem 9.2 and Theorem 9.11], the Koszul dual $\operatorname{Ext}^*_\Lambda(k, k)$ of Λ is the polynomial ring $k[x_1, \dots, x_n]$, and via the map

$$- \otimes_\Lambda k \colon \operatorname{HH}^*(\Lambda) \to \operatorname{Ext}^*_\Lambda(k, k)$$

this is a finitely generated $\operatorname{HH}^*(\Lambda)$-module. By choosing H to be the even part of the inverse image of $k[x_1, \dots, x_n]$, we see that **Fg** holds.

Having pointed out these three examples where **Fg** hold, we now proceed with the following result. Together with the Ghost Lemma, it is the key ingredient in the proof of the main result.

Proposition 3.1. *Suppose Λ is selfinjective and* **Fg** *holds. Then for every Λ-module M such that $\operatorname{cx} M \geq 2$, there exists a sequence*

$$M = K_0 \xrightarrow{f_1} K_1 \to \cdots \xrightarrow{f_{c-1}} K_{c-1}$$

of modules and monomorphisms in $\operatorname{mod} \Lambda$*, where $c = \operatorname{cx} M$, satisfying the following:*

(i) $\operatorname{cx} K_j = c - j$ *for each* $0 \leq j \leq c - 1$,

(ii) *for each* $1 \leq j \leq c-1$ *the map* $\operatorname{Ext}^i_\Lambda(K_j, M) \xrightarrow{(f_j)^*} \operatorname{Ext}^i_\Lambda(K_{j-1}, M)$ *vanishes for* $i \gg 0$,

(iii) *the composition* $f_{c-1} \circ \cdots \circ f_1$ *is nonzero in* $\underline{\operatorname{mod}}\Lambda$.

Proof. First we recall some general facts. Let R be a positively graded commutative Noetherian k-algebra of finite type, and let $G = \oplus_{n=0}^{\infty} G^n$ be a finitely generated graded R-module. An application of the prime avoidance lemma gives the existence of a homogeneous element $r \in R$, of positive degree, such that the multiplication map

$$G^n \xrightarrow{\cdot r} G^{n+|r|}$$

is injective for $n \gg 0$ (cf. [Be1, Proposition 2.1]). Therefore, since **Fg** holds for Λ, given any Λ-modules X and Y there exists a homogeneous element $\eta \in H^+$ such that the multiplication map

$$\operatorname{Ext}_{\Lambda}^{i}(X,Y) \xrightarrow{\cdot \eta} \operatorname{Ext}_{\Lambda}^{i+|\eta|}(X,Y)$$

is injective for $i \gg 0$.

Choose a homogeneous element $\eta_1 \in H^+$ such that the multiplication map

$$\operatorname{Ext}_{\Lambda}^{i}(M, M \oplus \Lambda/\mathfrak{r}) \xrightarrow{\cdot \eta_1} \operatorname{Ext}_{\Lambda}^{i+|\eta_1|}(M, M \oplus \Lambda/\mathfrak{r})$$

is injective for $i \gg 0$. Applying the map ϕ_M to η_1 gives a short exact sequence

$$\phi_M(\eta_1)\colon 0 \to M \xrightarrow{f_1} K_1 \to \Omega_{\Lambda}^{|\eta_1|-1}(M) \to 0,$$

and the arguments used in the proof of [Be1, Theorem 3.2] shows that $\operatorname{cx} K_1 = c-1$. Next, if $c \geq 3$, choose a homogeneous element $\eta_2 \in H^+$ such that the multiplication map

$$\operatorname{Ext}_{\Lambda}^{i}(K_1, M \oplus \Lambda/\mathfrak{r}) \oplus \operatorname{Ext}_{\Lambda}^{i}(M, K_1) \xrightarrow{\cdot \eta_2} \operatorname{Ext}_{\Lambda}^{i+|\eta_2|}(K_1, M \oplus \Lambda/\mathfrak{r}) \oplus \operatorname{Ext}_{\Lambda}^{i+|\eta_2|}(M, K_1)$$

is injective for $i \gg 0$. Applying the map ϕ_{K_1} to η_2 gives a short exact sequence

$$\phi_{K_1}(\eta_2)\colon 0 \to K_1 \xrightarrow{f_2} K_2 \to \Omega_{\Lambda}^{|\eta_2|-1}(K_1) \to 0,$$

in which $\operatorname{cx} K_2 = c-2$. We continue this process until we end up with a module K_{c-1} of complexity 1. Thus we obtain homogeneous elements $\eta_1, \dots, \eta_{c-1} \in H^+$, and for each $1 \leq j \leq c-1$ a short exact sequence

$$\phi_{K_{j-1}}(\eta_j)\colon 0 \to K_{j-1} \xrightarrow{f_j} K_j \to \Omega_{\Lambda}^{|\eta_j|-1}(K_{j-1}) \to 0$$

with $\operatorname{cx} K_j = c-j$ (here $K_0 = M$). For each j the element η_j is chosen in such a way that it is regular on $\operatorname{Ext}_{\Lambda}^{i}(K_{j-1}, M \oplus \Lambda/\mathfrak{r}) \oplus \operatorname{Ext}_{\Lambda}^{i}(M, K_{j-1})$ for $i \gg 0$.

We show that the sequence

$$M \xrightarrow{f_1} K_1 \to \cdots \xrightarrow{f_{c-1}} K_{c-1}$$

of modules and maps satisfies (i), (ii) and (iii) in the statement of the proposition. It follows directly from the construction that $\operatorname{cx} K_j = c-j$, hence (i) holds. Now for each $1 \leq j \leq c-1$ and $i \gg 0$, the exact sequence $\phi_{K_{j-1}}(\eta_j)$ induces the two exact sequences

$$\operatorname{Ext}_{\Lambda}^{i}(K_j, M) \xrightarrow{(f_j)^*} \operatorname{Ext}_{\Lambda}^{i}(K_{j-1}, M) \xrightarrow{\cdot \eta_j} \operatorname{Ext}_{\Lambda}^{i+|\eta_j|}(K_{j-1}, M)$$

and

$$\operatorname{Ext}_{\Lambda}^{i}(M, K_{j-1}) \xrightarrow{(f_j)_*} \operatorname{Ext}_{\Lambda}^{i}(M, K_j) \longrightarrow \operatorname{Ext}_{\Lambda}^{i-|\eta_j|+1}(M, K_{j-1}) \xrightarrow{\cdot \eta_j} \operatorname{Ext}_{\Lambda}^{i+1}(M, K_{j-1}).$$

From the upper exact sequence, we see that the map $\operatorname{Ext}_{\Lambda}^{i}(K_j, M) \xrightarrow{(f_j)^*} \operatorname{Ext}_{\Lambda}^{i}(K_{j-1}, M)$ vanishes for $i \gg 0$, since the multiplication map involving η_j is injective. Consequently (ii) holds. From the lower exact sequence we see that the map $\operatorname{Ext}^i_\Lambda(M, K_{j-1}) \xrightarrow{(f_j)_*} \operatorname{Ext}^i_\Lambda(M, K_j)$ is surjective for $i \gg 0$. This implies that when i is large, the maps $f_1, \dots, f_{c-1}$ induce a chain

$$\operatorname{Ext}^i_\Lambda(M, M) \xrightarrow{(f_1)_*} \operatorname{Ext}^i_\Lambda(M, K_1) \to \cdots \xrightarrow{(f_{c-1})_*} \operatorname{Ext}^i_\Lambda(M, K_{c-1})$$

of epimorphisms. Now choose a homogeneous element $\eta \in H^+$ which is regular on $\operatorname{Ext}^i_\Lambda(K_{c-1}, \Lambda/\mathfrak{r})$ for $i \gg 0$. Applying $\phi_{K_{c-1}}$ to this element gives an element

$$\phi_{K_{c-1}}(\eta) \colon 0 \to K_{c-1} \to K \to \Omega_\Lambda^{|\eta|-1}(K_{c-1}) \to 0$$

in $\operatorname{Ext}^*_\Lambda(K_{c-1}, K_{c-1})$, where the module K is projective. Using the arguments in the proof of [Be2, Corollary 3.2], we see that $\phi_{K_{c-1}}(\eta)$ cannot be nilpotent in $\operatorname{Ext}^*_\Lambda(K_{c-1}, K_{c-1})$. Consequently, given any $w \in \mathbb{N}$, there is an integer $i \geq w$ such that $\operatorname{Ext}^i_\Lambda(K_{c-1}, K_{c-1})$ is nonzero. Using the exact sequences $\phi_M(\eta_1), \phi_{K_1}(\eta_2), \dots, \phi_{K_{c-2}}(\eta_{c-1})$ we then see that given any $w \in \mathbb{N}$, there is an integer $i \geq w$ such that $\operatorname{Ext}^i_\Lambda(M, K_{c-1})$ is nonzero. This shows that the composition

$$M \xrightarrow{f_1} K_1 \to \cdots \xrightarrow{f_{c-1}} K_{c-1}$$

is nonzero in $\underline{\operatorname{mod}}\Lambda$, and so (iii) holds. $\square$

We now prove the main result: when Λ is selfinjective and **Fg** holds, then the dimension of the stable module category of Λ is at least $\operatorname{cx} \Lambda/\mathfrak{r} - 1$.

Theorem 3.2. *If Λ is selfinjective and* **Fg** *holds, then* $\dim(\underline{\operatorname{mod}}\Lambda) \geq \operatorname{cx} \Lambda/\mathfrak{r} - 1$.

Proof. Denote the complexity of $\Lambda/\mathfrak{r}$ by c. Let $M \in \operatorname{mod} \Lambda$ be a module generating $\underline{\operatorname{mod}}\Lambda$, i.e. there exists a number n such that $\langle M \rangle_n = \underline{\operatorname{mod}}\Lambda$. Our aim is to show that $n \geq c$. If $c \leq 1$, then there is nothing to prove, so we may assume $c \geq 2$.

By Lemma 2.3 the module M must have maximal complexity, that is, the equality $\operatorname{cx} M = c$ holds. Choose a sequence

$$M = K_0 \xrightarrow{f_1} K_1 \to \cdots \xrightarrow{f_{c-1}} K_{c-1}$$

of modules and maps as in Proposition 3.1, and consider the functors $\underline{\operatorname{Hom}}_\Lambda(K_j, -)$ on $\underline{\operatorname{mod}}\Lambda$, together with the natural transformations

$$\underline{\operatorname{Hom}}_\Lambda(K_{c-1}, -) \xrightarrow{(f_{c-1})^*} \underline{\operatorname{Hom}}_\Lambda(K_{c-2}, -) \to \cdots \xrightarrow{(f_1)^*} \underline{\operatorname{Hom}}_\Lambda(M, -).$$

Since the map $\operatorname{Ext}^i_\Lambda(K_j, M) \xrightarrow{(f_j)^*} \operatorname{Ext}^i_\Lambda(K_{j-1}, M)$ vanishes for $i \gg 0$, the map $\underline{\operatorname{Hom}}_\Lambda(K_j, \Omega^i_\Lambda(M)) \xrightarrow{(f_j)^*} \underline{\operatorname{Hom}}_\Lambda(K_{j-1}, \Omega^i_\Lambda(M))$ vanishes for $i \ll 0$. From Lemma 2.1 we conclude that for every module $X \in \langle M \rangle_{c-1}$, the map

$$\underline{\operatorname{Hom}}_\Lambda(K_{c-1}, \Omega^i_\Lambda(X)) \xrightarrow{(f_{c-1} \circ \cdots \circ f_1)^*} \underline{\operatorname{Hom}}_\Lambda(M, \Omega^i_\Lambda(X))$$

vanishes for $i \ll 0$. However, by [Be1, Theorem 2.3] the module K_{c-1} is periodic in $\underline{\operatorname{mod}}\Lambda$, that is, there is an integer $p \geq 1$ such that $K_{c-1} \simeq \Omega^p_\Lambda(K_{c-1})$ in $\underline{\operatorname{mod}}\Lambda$. Therefore, since the composition $f_{c-1} \circ \cdots \circ f_1$ is nonzero in $\underline{\operatorname{mod}}\Lambda$, the map

$$\underline{\operatorname{Hom}}_\Lambda(K_{c-1}, \Omega^{ip}_\Lambda(K_{c-1})) \xrightarrow{(f_{c-1} \circ \cdots \circ f_1)^*} \underline{\operatorname{Hom}}_\Lambda(M, \Omega^{ip}_\Lambda(K_{c-1}))$$

does not vanish for any $i \in \mathbb{Z}$. This shows that the module K_{c-1} cannot be an element in $X \in \langle M \rangle_{c-1}$, and so $n \geq c$. $\square$

Using Proposition 2.2 and Auslander's upper bound, we obtain the promised result on the representation dimension. We denote by $\ell\ell(\Lambda)$ the Loewy length of our algebra Λ.

Theorem 3.3. *If Λ is a non-semisimple selfinjective algebra and* **Fg** *holds, then*

$$\operatorname{cx} \Lambda/\mathfrak{r} + 1 \leq \operatorname{repdim} \Lambda \leq \ell\ell(\Lambda).$$

Note that, in particular, this theorem shows that the Loewy length of a selfinjective Artin algebra satisfying **Fg** is strictly larger than the maximal complexity occurring among the modules. This is an analogue of Benson's conjecture, which states that the Loewy length of a block of a group algebra in characteristic p is strictly larger than the p-rank of its defect group. The characteristic 2 case of Benson's conjecture was proved by Rouquier in [Ro2], whereas the general case was proved by Oppermann in [Op1].

Rouquier showed that the representation dimension of the exterior algebra on an n-dimensional vector space is exactly $n+1$. It therefore seems natural to ask the following:

Question. When **Fg** holds, what is the exact value of $\operatorname{repdim} \Lambda$?

The following corollaries to Theorem 3.3 provide lower bounds for the representation dimension of the algebras given in the three examples prior to Theorem 3.2. In particular, we obtain [Op1, Corollary 19], half of [Ro2, Theorem 4.1] and the result of Avramov and Iyengar on the representation dimension of Artin complete intersections (cf. [AvI]).

Corollary 3.4. *Suppose k is a field of positive characteristic p, and let G be a finite group whose order is divisible by p. Then* $\operatorname{repdim} kG \geq p-\operatorname{rank} G + 1$, *that is, the representation dimension of kG is strictly greater than* $\operatorname{Krulldim} \operatorname{H}^*(G,k)$.

Proof. By a result of Quillen (cf. [Qu1, Qu2]), the complexity of the trivial kG-module, i.e. the Krull dimension of the cohomology ring $\operatorname{H}^*(G,k)$, equals the p-rank of G. □

Corollary 3.5. *Let A be a commutative Noetherian local complete intersection of codimension c. If A is Artin, then* $\operatorname{repdim} A \geq c+1$.

Proof. By a classical result of Tate (cf. [Tat, Theorem 6]), the complexity of the simple module over a complete intersection equals the codimension of the ring. □

Remark. Let A be a commutative Noetherian local complete intersection of codimension c. If A is complete, then the proof of Theorem 3.2 also applies to the stable category of finitely generated maximal Cohen-Macaulay A-modules. Namely, the dimension of this triangulated category is at least $c-1$.

Corollary 3.6. *Suppose k is a field and that Λ is a finite dimensional non-semisimple selfinjective k-algebra, with the property that $\Lambda/\mathfrak{r} \otimes_k \Lambda/\mathfrak{r}$ is semisimple (as happens for example when k is algebraically closed). Furthermore, suppose the Hochschild cohomology ring* $\operatorname{HH}^*(\Lambda)$ *is Noetherian, and that* $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ *is a finitely generated* $\operatorname{HH}^*(\Lambda)$*-module. Then* $\operatorname{repdim} \Lambda \geq \operatorname{Krulldim} \operatorname{HH}^*(\Lambda)+1$.

Proof. The Krull dimension of $\operatorname{HH}^*(\Lambda)$ is its rate of growth $\gamma(\operatorname{HH}^*(\Lambda))$ as a graded k-vector space. Therefore, since the $\operatorname{HH}^*(\Lambda)$-module $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ is finitely generated, we see that

$$\operatorname{cx} \Lambda/\mathfrak{r} = \gamma\left(\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})\right) \leq \gamma\left(\operatorname{HH}^*(\Lambda)\right) = \operatorname{Krulldim} \operatorname{HH}^*(\Lambda).$$

Denote the radical of Λ^{e} by $\mathfrak{r}^{\mathrm{e}}$, and let B be a finitely generated bimodule (i.e. $B \in \operatorname{mod} \Lambda^{\mathrm{e}}$). If B is not simple, then choose an exact sequence

$$0 \to S \to B \to B' \to 0$$

in which S is simple. This sequence induces an exact sequence

$$\operatorname{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda,S) \to \operatorname{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda,B) \to \operatorname{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda,B')$$

of $\mathrm{HH}^*(\Lambda)$-modules, all of which are finitely generated by [EHSST, Proposition 2.4]. Consequently the inequality

$$\gamma\left(\mathrm{Ext}^*_{\Lambda^e}(\Lambda, B)\right) \le \max\{\gamma\left(\mathrm{Ext}^*_{\Lambda^e}(\Lambda, S)\right), \gamma\left(\mathrm{Ext}^*_{\Lambda^e}(\Lambda, B')\right)\}$$

holds, and so induction on length gives

$$\gamma\left(\mathrm{Ext}^*_{\Lambda^e}(\Lambda, B)\right) \le \gamma\left(\mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda^e/\mathfrak{r}^e)\right) = \mathrm{cx}_{\Lambda^e} \Lambda.$$

In particular, the inequality $\gamma\left(\mathrm{HH}^*(\Lambda)\right) \le \mathrm{cx}_{\Lambda^e} \Lambda$ holds. But the complexity of Λ as a bimodule equals that of the Λ-module $\Lambda/\mathfrak{r}$. Namely, applying $-\otimes_\Lambda \Lambda/\mathfrak{r}$ to the minimal projective bimodule resolution of Λ gives the minimal Λ-projective resolution of $\Lambda/\mathfrak{r}$. Therefore $\mathrm{cx}_{\Lambda^e} \Lambda = \mathrm{cx}\, \Lambda/\mathfrak{r}$, and this shows that the Krull dimension of $\mathrm{HH}^*(\Lambda)$ equals $\mathrm{cx}\, \Lambda/\mathfrak{r}$. □

We end this paper with two examples illustrating the main results. These provide new examples of classes of algebras with arbitrarily large representation dimension. In the first example we generalize Rouquier's result on exterior algebras. Namely, we determine the exact value of the representation dimension of certain quantum exterior algebras.

Examples. (i) Let k be a field, let $n \ge 1$ be an integer, and let Λ be the quantum exterior algebra

$$k\langle X_1, \dots, X_n\rangle/(X_i^2, \{X_iX_j - q_{ij}X_jX_i\}_{i<j}),$$

where $0 \ne q_{ij} \in k$. This algebra is finite dimensional of dimension 2^n, and the complexity of k is n. Furthermore, this is a Frobenius algebra; the codimension two argument in the beginning of [BeE, Section 3] carries over. In particular, this algebra is selfinjective, and it was shown in [ErS] that **Fg** holds if and only if all the q_{ij} are roots of unity. Therefore, when this is the case, then the representation dimension of Λ is at least $n+1$. However, the Loewy length of Λ is $n+1$, and by Auslander's result this is an upper bound of the representation dimension. Consequently, when all the q_{ij} are roots of unity, then $\mathrm{repdim}\, \Lambda = n+1$.

(ii) Let k be an algebraically closed field, and let R be a Noetherian Artin-Schelter regular Koszul k-algebra of dimension d. That is, R is graded connected of global dimension d, its Gelfand-Kirillov dimension is finite, and

$$\mathrm{Ext}^i_R(k, R) \simeq \begin{cases} 0 & i \ne d \\ k & i = d \text{ (up to shift).} \end{cases}$$

If R is a finitely generated module over its center, then by [Sol, Proposition 9.15] the Koszul dual Λ of R is selfinjective, satisfies **Fg**, and $\mathrm{cx}\, \Lambda/\mathfrak{r} = d$. Thus in this case the representation dimension of Λ is at least $d+1$.

An example of such an algebra is obtained from the Sklyanin algebras (cf. [Smi, Section 8]: let E be an elliptic curve over k, and fix a point $P \in E$ such that $nP = 0$ for some $n \ge 1$. Denote by $\sigma_P\colon E \to E$ the corresponding translation automorphism. Furthermore, let $d \ge 1$ be an integer, and let $A_d(E, \sigma_P)$ be the d-dimensional Sklyanin algebra. This is an Artin-Schelter regular algebra of the above type.

References

[Au1] M. Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes, London, 1971, republished in [Au2].

[Au2] M. Auslander, *Selected works of Maurice Auslander. Part 1*, I. Reiten, S. Smalø, Ø. Solberg (editors), Amer. Math. Soc., Providence, 1999.

[Avr] L. Avramov, *Modules of finite virtual projective dimension*, Invent. Math. 96 (1989), 71-101.

[AvI] L. Avramov, S. Iyengar, *The dimension of the stable derived category of a complete intersection local ring*, in preparation.

[Be1] P. Bergh, *Complexity and periodicity*, Colloq. Math. 104 (2006), no. 2, 169-191.
[Be2] P. Bergh, *Modules with reducible complexity*, J. Algebra 310 (2007), 132-147.
[BeE] P. Bergh, K. Erdmann, *Homology and cohomology of quantum complete intersections*, Algebra Number Theory 2 (2008), no. 5, 501-522.
[BeO] P. Bergh, S. Oppermann, *The representation dimension of quantum complete intersections*, J. Algebra 320 (2008), no. 1, 354-368.
[EHSST] K. Erdmann, M. Holloway, N. Snashall, Ø. Solberg, R. Taillefer, *Support varieties for selfinjective algebras*, K-theory 33 (2004), 67-87.
[ErS] K. Erdmann, Ø. Solberg, *Finite generation of the Hochschild cohomology ring of some Koszul algebras*, in preparation.
[Eve] L. Evens, *The cohomology ring of a finite group*, Trans. Amer. Math. Soc. 101 (1961), 224-239.
[IgT] K. Igusa, G. Todorov, *On the finitistic global dimension conjecture for Artin algebras*, in *Representations of algebras and related topics*, 201-204, Fields Inst. Commun. 45, Amer. Math. Soc., Providence, 2005.
[Iya] O. Iyama, *Finitness of representation dimension*, Proc. Amer. Math. Soc. 131 (2003), no. 4, 1011-1014.
[KrK] H. Krause, D. Kussin, *Rouquier's theorem on representation dimension*, in *Trends in representation theory of algebras and related topics*, 95-103, Contemp. Math. 406, Amer. Math. Soc., Providence, 2006.
[Op1] S. Oppermann, *A lower bound for the representation dimension of* kC_p^n, Math. Z. 256 (2007), no. 3, 481-490.
[Op2] S. Oppermann, *Lower bounds for Auslander's representation dimension*, preprint.
[Qu1] D. Quillen, *The spectrum of an equivariant cohomology ring, I*, Ann. of Math. 94 (1971), 549-572.
[Qu2] D. Quillen, *The spectrum of an equivariant cohomology ring, II*, Ann. of Math. 94 (1971), 573-602.
[Ro1] R. Rouquier, *Dimensions of triangulated categories*, J. K-theory 1 (2008), no. 2, 193-256.
[Ro2] R. Rouquier, *Representation dimension of exterior algebras*, Invent. Math. 165 (2006), no. 2, 357-367.
[Smi] P. Smith, *Some finite-dimensional algebras related to elliptic curves*, in *Representation theory of algebras and related topics (Mexico City, 1994)*, 315-348, CMS Conf. Proc. 19, Amer. Math. Soc., Providence, 1996.
[Sol] Ø. Solberg, *Support varieties for modules and complexes*, in *Trends in representation theory of algebras and related topics*, 239-270, Contemp. Math. 406, Amer. Math. Soc., Providence, 2006.
[Tat] J. Tate, *Homology of noetherian rings and local rings*, Ill. J. Math. 1 (1957), 14-27.
[Yon] N. Yoneda, *Note on products in* Ext, Proc. Amer. Math. Soc. 9 (1958), 873-875.

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