

HOMOLOGICAL ALGEBRA MODULO EXACT ZERO-DIVISORS

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ABSTRACT. We study the homological behavior of modules over local rings modulo exact zero-divisors. We obtain new results which are in some sense “opposite” to those known for modules over local rings modulo regular elements.

1. INTRODUCTION

Given a local (meaning also commutative and Noetherian) ring S and an ideal \mathfrak{a} , one may ask whether the homological behavior of modules over S/\mathfrak{a} is related to that over S . In general this is hopeless; one needs to restrict the ideal \mathfrak{a} . When \mathfrak{a} is generated by a regular sequence, there is a well developed and powerful theory relating the homological properties of modules over these two rings, cf. [Av1], [Eis] and [Gu1]. As an illustration, suppose that the ideal \mathfrak{a} is generated by a single regular element x , denote the factor ring $S/(x)$ by R , and let M and N be R -modules. The sequence

$$0 \rightarrow S \xrightarrow{x} S \rightarrow R \rightarrow 0$$

of S -modules is exact and therefore $0 \rightarrow S \xrightarrow{x} S \rightarrow 0$ is a projective resolution of R over S . Tensoring this resolution with N and bearing in mind that $xN = 0$, we see $\mathrm{Tor}_0^S(R, N) \cong \mathrm{Tor}_1^S(R, N) \cong N$ and $\mathrm{Tor}_q^S(R, N) = 0$ for $q \geq 2$. Consequently the E^2 -page of the first quadrant change of rings spectral sequence

$$\mathrm{Tor}_p^R(M, \mathrm{Tor}_q^S(R, N)) \underset{p}{\implies} \mathrm{Tor}_{p+q}^S(M, N)$$

contains only two nonzero rows and therefore it collapses to the long exact sequence

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & & & & & \\ \mathrm{Tor}_2^R(M, N) & \rightarrow & \mathrm{Tor}_3^S(M, N) & \rightarrow & \mathrm{Tor}_3^R(M, N) & \rightarrow & \\ & & & & & & \\ \mathrm{Tor}_1^R(M, N) & \rightarrow & \mathrm{Tor}_2^S(M, N) & \rightarrow & \mathrm{Tor}_2^R(M, N) & \rightarrow & \\ & & & & & & \\ M \otimes_R N & \rightarrow & \mathrm{Tor}_1^S(M, N) & \rightarrow & \mathrm{Tor}_1^R(M, N) & \rightarrow & 0 \end{array}$$

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in homology. From this exact sequence a number of results may be deduced. One such result that concerns the vanishing of homology is that if

$$\mathrm{Tor}_1^R(M, N) = \mathrm{Tor}_2^R(M, N) = \cdots = \mathrm{Tor}_n^R(M, N) = 0$$

for some $n \geq 2$, then

$$\mathrm{Tor}_2^S(M, N) = \cdots = \mathrm{Tor}_n^S(M, N) = 0.$$

In other words, vanishing of homology over R implies the vanishing of homology over S . An analogous result for cohomology is obtained by using the third quadrant change of rings spectral sequence

$$\mathrm{Ext}_R^p(M, \mathrm{Ext}_S^q(R, N)) \Longrightarrow \mathrm{Ext}_S^{p+q}(M, N),$$

which also collapses to a long exact sequence (cf. for example [Be], [Jo], [AvBu] or [Dao] for detailed discussions of the vanishing of (co)homology over local rings modulo regular sequences.)

Another result, obtained from the long exact sequence discussed above, compares the complexity of an R -module with its complexity as an S -module. Namely there are inequalities [Av1, 3.2(3)]:

$$\mathrm{cx}_S(M) \leq \mathrm{cx}_R(M) \leq \mathrm{cx}_S(M) + 1$$

In this paper we study the case where the element $x \in S$ is in some sense the “next best thing” to being a regular element. More precisely, we consider the case where the annihilator of x is a nonzero principal ideal whose annihilator is also principal (and therefore is the ideal (x)). Following [HeS], the element x is said to be an *exact zero-divisor* if it is nonzero, belongs to the maximal ideal of S , and there exists another element $y \in S$ such that $\mathrm{ann}_S(x) = (y)$ and $\mathrm{ann}_S(y) = (x)$. In this case we say that (x, y) is a *pair of exact zero-divisors* of S . Furthermore, the ideal (x) is then an example of a *quasi-complete intersection ideal*, a notion introduced in [AHS]. In that same paper various results relating certain invariants of modules over S with those over $S/(x)$ are proved. We continue along similar lines and show that even if the element x is the next best thing to being regular, namely an exact zero-divisor, the homological relationships between modules over $S/(x)$ and $S/(x)$ -modules over S change dramatically compared to the case where x is regular. Two of our main results in Section 2 concern the vanishing of (co)homology. In particular the result for homology takes the following form:

Theorem. Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S . Furthermore, let M and N be R -modules such that $yN = 0$. If there exists an integer $n \geq 2$ such that $\mathrm{Tor}_i^R(M, N) = 0$ for $1 \leq i \leq n$, then $\mathrm{Tor}_i^S(M, N) \cong M \otimes_S N$ for $1 \leq i \leq n - 1$.

Compared to the rigidity result for the case where x is regular, the conclusion of the previous theorem is opposite in the sense that it is an anti-rigidity result: vanishing of homology over R implies the *non-vanishing* of homology over S (when the modules involved are nonzero and finitely generated.) For cohomology, we obtain the following analogue:

Theorem. Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S . Furthermore, let M and N be R -modules such that $yN = 0$. If there exists an integer $n \geq 2$ such that $\mathrm{Ext}_R^i(M, N) = 0$ for $1 \leq i \leq n$, then $\mathrm{Ext}_S^i(M, N) \cong \mathrm{Hom}_S(M, N)$ for $1 \leq i \leq n - 1$.

We end Section 2 with some applications of these vanishing results to the study of depth and dimension of Tor modules.

We also compare the complexities of finitely generated modules over R and over S (cf. Section 3 for the definition of complexity). Similar to our previous results, we show that such a comparison is quite different than the case where x is regular. The following theorem is the main result of section 3.

Theorem. Let $R = S/(x)$ where S is a local ring and x is an exact zero-divisor in S . If M is a finitely generated R -module, then for any n there are inequalities

$$\beta_n^R(M) - \sum_{i=0}^{n-2} \beta_i^R(M) \leq \beta_n^S(M) \leq \sum_{i=0}^n \beta_i^R(M)$$

of Betti-numbers. In particular, the inequality $\text{cx}_S(M) \leq \text{cx}_R(M) + 1$ holds.

In the final section, Section 4, we discuss canonical endomorphisms of complexes of finitely generated free R -modules, and canonical elements of $\text{Ext}_R^2(M, M)$, for finitely generated R -modules M , in the case where $R = S/(x)$ and (x, x) is a pair of exact zero-divisors of S . The main result, Theorem 4.6, equates the ability to lift a finitely generated R -module M from R to S to the triviality of the canonical element in $\text{Ext}_R^2(M, M)$. This generalizes results, which seem to be folklore, on lifting modules from $T/(x)$ to $T/(x^2)$ in the case where x is a non-zero-divisor of the local ring T (cf. Example 4.7 below).

2. VANISHING RESULTS

In this section we prove our vanishing results, starting with the homology version. We fix a local ring S , a pair of exact zero-divisors (x, y) , and denote the local ring $S/(x)$ by R . The proof of the homology vanishing result uses the same change of rings spectral sequence stated in the introduction. Thus, for an R -module N , we need to compute the homology groups $\text{Tor}_q^S(R, N)$. This is elementary: the periodic complex

$$\cdots \rightarrow S \xrightarrow{y} S \xrightarrow{x} S \xrightarrow{y} S \xrightarrow{x} S \rightarrow 0$$

is the minimal projective resolution of R over S . The same sequence makes it easy to compute the cohomology groups $\text{Ext}_S^q(R, N)$ which we will need for the proof of the cohomology version of our theorem. We record this observation in the next lemma.

Lemma 2.1. *Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S . Then, for every R -module N ,*

$$\text{Tor}_q^S(R, N) \cong \begin{cases} N & \text{for } q = 0 \\ N/yN & \text{for } q > 0 \text{ odd} \\ \text{ann}_N(y) & \text{for } q > 0 \text{ even} \end{cases}$$

and

$$\text{Ext}_S^q(R, N) \cong \begin{cases} N & \text{for } q = 0 \\ \text{ann}_N(y) & \text{for } q > 0 \text{ odd} \\ N/yN & \text{for } q > 0 \text{ even} \end{cases}$$

In particular, if $yN = 0$, then

$$\text{Tor}_q^S(R, N) \cong N \cong \text{Ext}_S^q(R, N)$$

for all $q \geq 0$.

Consequently, columns 1 through n all vanish, i.e., $E_{p,q}^2 = 0$ for all $q \in \mathbb{Z}$ and $1 \leq p \leq n$. Fixing such p and q , we see that $E_{p,q}^\infty$ also vanishes since this term is a subquotient of $E_{p,q}^2$. Letting H_i denote $\text{Tor}_i^S(M, N)$ for all i , we have a filtration $\{\Phi^j H_i\}$ of H_i satisfying

$$0 = \Phi^{-1}H_i \subseteq \Phi^0H_i \subseteq \dots \subseteq \Phi^{i-1}H_i \subseteq \Phi^iH_i = H_i,$$

with $E_{j,i-j}^\infty \cong \Phi^j H_i / \Phi^{j-1} H_i$ for all i and j . Thus the vanishing of $E_{p,q}^\infty$ implies that $\Phi^p H_{p+q} = \Phi^{p-1} H_{p+q}$, that is, $\Phi^p H_q = \Phi^{p-1} H_q$ for all $q \in \mathbb{Z}$ and $1 \leq p \leq n$.

Now consider the zeroth column of the E^2 -page. For a positive q , the $E_{0,q}^2$ -term is isomorphic to $M \otimes_R N/yN$ when q is odd, and isomorphic to $M \otimes_R \text{ann}_N(y)$ when q is even. Since $E_{p,q}^2 = 0$ for all $q \in \mathbb{Z}$ and $1 \leq p \leq n$, there is an isomorphism $E_{0,q}^\infty \cong E_{0,q}^2$ for $q \leq n-1$, giving

$$E_{0,q}^\infty \cong \begin{cases} M \otimes_R N/yN & \text{for } 0 < q < n \text{ and } q \text{ odd} \\ M \otimes_R \text{ann}_N(y) & \text{for } 0 < q < n \text{ and } q \text{ even.} \end{cases}$$

But it follows from above that the equalities

$$E_{0,q}^\infty \cong \Phi^0 H_q = \Phi^1 H_q = \dots = \Phi^q H_q$$

hold when $q < n$. Therefore, since $\Phi^q H_q = \text{Tor}_q^S(M, N)$, we are done. \square

As an immediate corollary we obtain the result, for the vanishing of homology, stated in the introduction. Note that, when $yN = 0$, N/yN and $\text{ann}_N(y)$ are both isomorphic to N .

Corollary 2.3. *Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S . Furthermore, let M and N be R -modules such that $yN = 0$. If there exists an integer $n \geq 2$ such that $\text{Tor}_i^R(M, N) = 0$ for $1 \leq i \leq n$, then $\text{Tor}_i^S(M, N) \cong M \otimes_S N$ for $1 \leq i \leq n-1$.*

When the modules involved are finitely generated and nonzero, their tensor product is also nonzero. This shows that the vanishing of homology over R implies the *non-vanishing* of homology over S .

Corollary 2.4. *Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S . Furthermore, let M and N be nonzero finitely generated R -modules such that $yN = 0$. If there exists an integer $n \geq 2$ such that $\text{Tor}_i^R(M, N) = 0$ for $1 \leq i \leq n$, then $\text{Tor}_i^S(M, N) \neq 0$ for $1 \leq i \leq n-1$.*

A special situation occurs in case $x = y$, that is, when (x, x) forms a pair of exact zero-divisors. When we are in this situation, the condition that yN vanishes follows automatically from the fact that N is an R -module. Consequently, for such an exact zero-divisor, results we obtained so far adds up to the following:

Corollary 2.5. *Let $R = S/(x)$ where S is a local ring and (x, x) is a pair of exact zero-divisors in S . Furthermore, let M and N be R -modules and suppose there exists an integer $n \geq 2$ such that $\text{Tor}_i^R(M, N) = 0$ for $1 \leq i \leq n$. Then $\text{Tor}_i^S(M, N) \cong M \otimes_S N$ for $1 \leq i \leq n-1$. In particular, if M and N are nonzero and finitely generated, then $\text{Tor}_i^S(M, N) \neq 0$ for $1 \leq i \leq n-1$.*

In certain cases we can show that the Tors over S cannot vanish irrespective of vanishing of the Tors over R .

Proposition 2.6. *Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors, both of which are minimal generators of the maximal ideal of S . Furthermore, let M and N be nonzero finitely generated R -modules such that $yN = 0$. Then $\text{Tor}_i^S(M, N) \neq 0$ for all $i \geq 0$.*

Proof. Consider a minimal free resolution of M over S :

$$F : \cdots \rightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \rightarrow 0$$

Letting m denote a minimal generator of M , we can define the homomorphism $f : S/(x) \rightarrow M$ sending $\bar{1}$ to m . Because x and y are minimal generators of the maximal ideal of S , we can lift this homomorphism to a chain map

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & S & \xrightarrow{y} & S & \xrightarrow{x} & S & \longrightarrow & S/(x) & \longrightarrow & 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ \cdots & \longrightarrow & F_2 & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

in such a way that each f_i is a split injection. Tensoring the entire diagram with N we get the commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & N & \xrightarrow{0} & N & \xrightarrow{0} & N & \xrightarrow{=} & N & \longrightarrow & 0 \\ & & \downarrow f_2 \otimes N & & \downarrow f_1 \otimes N & & \downarrow f_0 \otimes N & & \downarrow f \otimes N & & \\ \cdots & \longrightarrow & F_2 \otimes_S N & \xrightarrow{\partial_2 \otimes N} & F_1 \otimes_S N & \xrightarrow{\partial_1 \otimes N} & F_0 \otimes_S N & \longrightarrow & M \otimes_S N & \longrightarrow & 0 \end{array}$$

in which the vertical maps are split injections. It follows that $\text{Tor}_i^S(M, N) \neq 0$ for all $i \geq 0$. \square

Next we prove the cohomological version of Theorem 2.2 without stating all the analogous corollaries. Recall that if A is a ring and X is a A -module, then $\text{add}_A X$ denotes the set of all direct summands of finite direct sums of copies of X .

Theorem 2.7. *Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S . Furthermore, let M and N be R -modules such that N/yN and $\text{ann}_N(y)$ both belong to $\text{add}_R N$. If there exists an integer $n \geq 2$ such that $\text{Ext}_R^i(M, N) = 0$ for $1 \leq i \leq n$, then*

$$\text{Ext}_S^i(M, N) \cong \begin{cases} \text{Hom}_S(M, \text{ann}_N(y)) & \text{for } 0 < i < n \text{ and } i \text{ odd} \\ \text{Hom}_S(M, N/yN) & \text{for } 0 < i < n \text{ and } i \text{ even} \end{cases}$$

Proof. Consider the E^2 -page of the third quadrant change of rings spectral sequence [Rot, Theorem 10.75]

$$\text{Ext}_R^p(M, \text{Ext}_S^q(R, N)) \Longrightarrow \text{Ext}_S^{p+q}(M, N)$$

$$\begin{array}{ccccccc}
 \cdots & & E_{-2,0}^2 & \xleftarrow{d_{0,-1}^2} & E_{-1,0}^2 & & E_{0,0}^2 \\
 & & & & & & \searrow \\
 \cdots & & E_{-2,-1}^2 & \xleftarrow{d_{0,-2}^2} & E_{-1,-1}^2 & & E_{0,-1}^2 \\
 & & & & & & \searrow \\
 \cdots & & E_{-2,-2}^2 & & E_{-1,-2}^2 & & E_{0,-2}^2 \\
 & & & & & & \vdots \\
 \vdots & & \vdots & & \vdots & & \vdots
 \end{array}$$

From Lemma 2.1, the term $E_{-p,-q}^2$ is given by

$$E_{-p,-q}^2 \cong \begin{cases} \text{Ext}_R^p(M, N) & \text{for } q = 0 \\ \text{Ext}_R^p(M, \text{ann}_N(y)) & \text{for } q > 0 \text{ odd} \\ \text{Ext}_R^p(M, N/yN) & \text{for } q > 0 \text{ even.} \end{cases}$$

Since N/yN and $\text{ann}_N(y)$ both belong to $\text{add}_R N$, the vanishing assumption on $\text{Ext}_R^i(M, N)$ implies that $\text{Ext}_R^i(M, \text{ann}_N(y)) = 0 = \text{Ext}_R^i(M, N/yN)$ for $1 \leq i \leq n$. Consequently, columns $-n$ through -1 of the E^2 -page all vanish, that is, $E_{p,q}^2 = 0$ for all $-n \leq p \leq -1$ and $q \in \mathbb{Z}$. Thus $E_{p,q}^\infty$ also vanishes for such p and q . Let H_{-i} denote $\text{Ext}_S^i(M, N)$ for all i . For each $i \geq 0$ we have a filtration $\{\Psi^j H_{-i}\}$ of H_{-i} satisfying

$$0 = \Psi^{-i-1} H_{-i} \subseteq \Psi^{-i} H_{-i} \subseteq \cdots \subseteq \Psi^{-1} H_{-i} \subseteq \Psi^0 H_{-i} = H_{-i},$$

where $E_{j,-i-j}^\infty \cong \Psi^j H_{-i} / \Psi^{j-1} H_{-i}$ for all i and j . For $-n \leq p \leq -1$ and all q , the vanishing of $E_{p,q}^\infty$ implies that $\Psi^p H_{p+q} = \Psi^{p-1} H_{p+q}$, that is, $\Psi^p H_q = \Psi^{p-1} H_q$ for all $q \in \mathbb{Z}$ and $-n \leq p \leq -1$.

Now consider the zeroth column of the E^2 -page. For negative q , the $E_{0,q}^2$ -term is isomorphic to $\text{Hom}_R(M, \text{ann}_N(y))$ when q is odd, and then isomorphic to $\text{Hom}_R(M, N/yN)$ when q is even. Since $E_{p,q}^2 = 0$ for all $q \in \mathbb{Z}$ and $-n \leq p \leq -1$, there is an isomorphism $E_{0,q}^\infty \cong E_{0,q}^2$ for $q \geq -n + 1$, giving

$$E_{0,q}^\infty \cong \begin{cases} \text{Hom}_R(M, \text{ann}_N(y)) & \text{for } -n < q < 0 \text{ and } q \text{ odd} \\ \text{Hom}_R(M, N/yN) & \text{for } -n < q < 0 \text{ and } q \text{ even.} \end{cases}$$

But it follows from above that the equalities

$$\Psi^{-1} H_q = \Psi^{-2} H_q = \cdots = \Psi^{-n-1} H_q = 0$$

hold when $q \geq -n$. Thus $E_{0,q}^\infty \cong \Psi^0 H_q = H_q = \text{Ext}_S^{-q}(M, N)$ for $-n < q < 0$ and the result follows. \square

Applications to depth and dimension. As applications of Theorem 2.2, we record some results that concern the depth and dimension of $\text{Tor}_i^S(M, N)$, and a formula involving the depths of M and N .

Note that for a finitely generated R -module M , $\text{depth}_R(M) = \text{depth}_S(M)$ and $\dim_R(M) = \dim_S(M)$ (cf. for example [BH]). We will use the convention that $\text{depth}(0) = \infty$ and call M a *maximal Cohen-Macaulay* R -module if its depth equals the dimension of R .

Corollary 2.8. *Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors. Let M and N be nonzero finitely generated R -modules such that $yN = 0$. Assume $\text{pd}_R(M) < \infty$ and that N is maximal Cohen-Macaulay. Then $\text{Tor}_i^S(M, N) \cong M \otimes N$ for all $i \geq 0$. In particular, if M is perfect (for example, if M is Cohen-Macaulay), then $\text{Tor}_i^S(M, N)$ is Cohen-Macaulay and $\dim(\text{Tor}_i^S(M, N)) = \dim(M)$ for all $i \geq 0$.*

Proof. It can be seen, by using induction on the dimension of R , that $\text{Tor}_i^R(M, N) = 0$ for all $i > 0$ (cf. [Yo, 2.2] or [Ce, 3.8]). Therefore the first claim follows from Corollary 2.3. Now if M is perfect, [Yo, 2.4] shows that $M \otimes N$ is Cohen-Macaulay with dimension equal to the dimension of M . This gives the required result. \square

Corollary 2.9. *Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors. Let M and N be nonzero finitely generated R -modules. Assume $yN = 0$ and $\text{depth}(N) \geq \text{pd}_R(M)$. Assume further that $\text{Tor}_i^R(M, N)$ has finite length for all $i > 0$. Then $\text{depth}(\text{Tor}_i^S(M, N)) = \text{depth}(N) - \text{pd}_R(M)$ for all $i \geq 0$.*

Proof. Let $q = \sup\{i : \text{Tor}_i^R(M, N) \neq 0\}$. It follows from [Au, 1.2] that $\text{depth}(M) + \text{depth}(N) = \text{depth}(R) - q$. Since the inequality $\text{depth}(M) + \text{depth}(N) \geq \text{depth}(R)$ holds by the Auslander-Buchsbaum formula, we have $q = 0$, and so $\text{Tor}_i^R(M, N) = 0$ for all $i > 0$. Now Corollary 2.3 and [Au, 1.2] show that $\text{depth}(\text{Tor}_i^S(M, N)) = \text{depth}(N) - \text{pd}_R(M)$ for all $i \geq 0$. \square

Huneke and Wiegand [HW, 2.5] prove that for a pair of finitely generated modules (M, N) over a complete intersection R , the vanishing of $\text{Tor}_i^R(M, N)$ for all positive i implies the equality

$$(DF) \quad \text{depth}(M) + \text{depth}(N) = \text{depth}(R) + \text{depth}(M \otimes N).$$

This remarkable equality, referred as the *depth formula*, was first proved by Auslander [Au] in the special case of vanishing of $\text{Tor}_i^R(M, N)$ for modules over an unramified regular local ring. The result of Huneke and Wiegand was generalized in [ArY] and [Iy] to modules of finite complete intersection dimension over arbitrary local rings (cf. also [BeJ]). Recall that a finitely generated module M over a local ring R is said to have *finite complete intersection dimension* [AGP], denoted by $\text{CI-dim}_R(M) < \infty$, if there is a diagram $R \rightarrow R' \leftarrow P$ of local homomorphisms where $R \rightarrow R'$ is flat, $R' \leftarrow P$ is surjective with kernel generated by a regular sequence of P contained in the maximal ideal of P , and $\text{pd}_P(M \otimes_R R') < \infty$. Standard examples of modules with finite complete intersection dimension are modules of finite projective dimension and those over complete intersections.

It is not known (at least to the authors) whether there exists a local ring R and a pair of finitely generated R -modules (M, N) such that $\text{Tor}_i^R(M, N) = 0$ for all $i > 0$ and the depth formula fails for (M, N) . If there is such a ring R and a pair of modules (M, N) with $\text{Tor}_i^R(M, N) = 0$ for all $i > 0$, then, by the above discussion, neither M nor N can have finite complete intersection dimension. Related to this fact, we observe in the next result, that over a quotient of a reduced Cohen-Macaulay local ring modulo an exact zero-divisor x , vanishing of $\text{Tor}_i^R(N, N)$ for all positive i with $yN = 0$ forces N to have infinite complete intersection dimension. This provides motivation to study the depth formula over local rings modulo exact zero divisors.

Over a ring A , the torsion submodule $\text{tr}(M)$ of an A -module M is defined to be the kernel of the natural map $M \rightarrow M \otimes_A K$ where K is the total quotient ring of A . If $\text{tr}(M) = M$, we say that M is a torsion A -module. Moreover, if M is a finitely generated torsion A -module, then it follows from the definition that there exists a regular element $a \in \mathfrak{m}$ such that $aM = 0$.

Proposition 2.10. *Let $R = S/(x)$ where S is a reduced local ring and (x, y) is a pair of exact zero-divisors in S . Let N be a nonzero finitely generated R -module such that $yN = 0$. Then N is a torsion R -module. Moreover, if S is Cohen-Macaulay and M is either a finite length R -module or $M = N$, then the depth formula over R_q fails for the pair (M_q, N_q) for some $q \in \text{Spec}(R)$. In addition, if $\text{Tor}_i^R(M, N) = 0$ for all $i > 0$, then $\text{CI-dim}_R(M) = \text{CI-dim}_R(N) = \infty$.*

Remark. The hypotheses in Proposition 2.10, that S is reduced and $yN = 0$, are necessary for the module N to be torsion: if S is not reduced, one can pick S to be an Artinian ring so that every module is vacuously torsion-free. Moreover, if $yN \neq 0$, then the module $N = R$ gives a counterexample over $R = S/(x)$ with $S = k[[X, Y]]/(XY)$.

Proof of Proposition 2.10. We have $\dim(S) > 0$, as S is reduced. Since $yN = 0$, we have $\text{Tor}_1^S(R, N) \cong N$ by Lemma 2.1. Therefore N is a torsion S -module, since S is reduced. Hence there exists an element $s \in \mathfrak{n}$ such that s is S -regular and $sN = 0$. Here \mathfrak{n} denotes the unique maximal ideal of S . Letting $r = s + (x) \in R$, we have $rN = 0$. Moreover, r is a regular element on R . To see this, assume that $rr' = 0$, where $r' \in R$. Choose $s' \in S$ such that $r' = s' + (x)$. Then $ss' \in (x) = \text{ann}_S(y)$. Hence $s(s'y) = (ss')y = 0$. Since s is a regular element on S , $s'y = 0$. Therefore $s' \in \text{ann}_S(y) = (x)$, and so $r' = 0$. This shows that N is a torsion R -module.

Next assume, to the contrary, that the depth formula (DF) holds for the pair (M_p, N_p) over R_p for all $p \in \text{Spec}(R)$. Then, as R is Cohen-Macaulay [AHŞ, 2.5 and 2.6], the R -module N satisfies Serre's condition (S_1) [EG, Chapter 3]. This implies N is torsion-free as an R -module, and hence contradicts the fact that $N \neq 0$. Thus the depth formula (DF) must fail for the pair (M_q, N_q) over R_q for some $q \in \text{Spec}(R)$.

Now assume that $\text{Tor}_i^R(M, N) = 0$ for all $i > 0$, and either $\text{CI-dim}_R(M) < \infty$ or $\text{CI-dim}_R(N) < \infty$, then either $\text{CI-dim}_{R_p}(M_p) < \infty$ or $\text{CI-dim}_{R_p}(N_p) < \infty$ for all $p \in \text{Spec}(R)$ [AGP, 1.6]. Since $\text{Tor}_i^{R_p}(M_p, N_p) = 0$ for all $p \in \text{Spec}(R)$, the depth formula holds for the pair (M_p, N_p) over R_p [ArY, 2.5], which contradicts the previous conclusion. Thus $\text{CI-dim}_R(M) = \text{CI-dim}_R(N) = \infty$. \square

Considering our previous result it seems reasonable to ask the following:

Question. Let $R = S/(x)$ where S is a reduced local Cohen-Macaulay ring and (x, y) is a pair of exact zero-divisors in S . Let M and N be non-zero finitely generated R -modules such that $yN = 0$. If $\text{Tor}_i^R(M, N) = 0$ for all $i > 0$, then do both M and N have infinite complete intersection dimension?

3. COMPLEXITY

As in the previous section, we fix a local ring S , a pair of exact zero-divisors (x, y) , and denote the local ring $S/(x)$ by R . In this section all modules are assumed to be finitely generated. Our aim is to compare free resolutions of modules over R with those over S and determine relationships involving complexities.

Given a local ring A and an A -module M , there exists a free resolution

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$$

which is minimal, that is, it appears as a direct summand of every free resolution of M . The cokernel of the map $F_{n+1} \rightarrow F_n$ is the n th *syzygy* module of M and denoted by $\Omega_A^n(M)$. Minimal free resolutions are unique up to isomorphisms and hence the syzygies are well-defined. Moreover, for every nonnegative integer n , the n th *Betti number* $\beta_n^A(M) \stackrel{\text{def}}{=} \text{rank } F_n$ is a well-defined invariant of M . It is well-known that $\dim_k \text{Ext}_A^n(M, k) = \beta_n^A(M) = \dim_k \text{Tor}_n^A(M, k)$ for every integer n where k is the residue field of A . It is also clear that the projective dimension of M is finite if and only if the Betti numbers of M eventually vanish. Thus the asymptotic behavior of the Betti sequence $\beta_0^A(M), \beta_1^A(M), \beta_2^A(M), \dots$ determines an important homological property of M . Following ideas from modular representation theory [Alp], an invariant measuring how “fast” the Betti sequence grows was introduced by Avramov in [Av1] (cf. also [Av2]). The *complexity* of M , denoted by $\text{cx}_A(M)$, is defined as

$$\text{cx}_A(M) \stackrel{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n^A(M) \leq an^{t-1} \text{ for all } n\},$$

and measures the polynomial rate of growth of the Betti sequence of M . It follows from the definition that M has finite projective dimension if and only if $\text{cx}_A(M) = 0$, whereas $\text{cx}_A(M) = 1$ if and only if the Betti sequence of M is bounded. For an arbitrary local ring, the complexity of a module is not necessarily finite [Av3, 4.2.2]. In fact, by [Gu2, Theorem 2.3], finiteness of complexity for all finitely generated A -modules is equivalent to A being a complete intersection.

We now return to our previous setting with exact zero-divisors and prove a lemma: every nonzero R -module has infinite projective dimension over S , that is, every such module has positive complexity over S .

Lemma 3.1. *Let $R = S/(x)$ where S is a local ring and x is an exact zero-divisor in S . Assume M is a nonzero finitely generated R -module. Then $\text{cx}_S(M) > 0$. In particular $\text{pd}_S(M) = \infty$.*

Proof. Suppose $\text{cx}_S(M) = 0$, that is, the module has finite projective dimension over S . Then $\text{Tor}_n^S(R, M) = 0$ for all higher n . But as we have seen in the proof of Lemma 2.1, these homology groups are those of the periodic complex

$$\cdots \rightarrow M \xrightarrow{y} M \xrightarrow{x} M \xrightarrow{y} M \xrightarrow{x} M \rightarrow 0,$$

in which the rightmost M is in degree zero. Therefore the vanishing of $\text{Tor}_n^S(R, M)$ for all higher n implies $\text{Tor}_n^S(R, M) = 0$ for all $n \geq 1$. Since $xM = 0$, vanishing of $\text{Tor}_1^S(R, M)$ shows $M = yM$. Thus $M = 0$ by Nakayama’s lemma. \square

Over a local ring A , the complexity of a module equals the complexity of any of its syzygies: their minimal free resolutions are the same except at the beginning. Moreover, given a short exact sequence

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

of A -modules, the inequality

$$(3.1) \quad \text{cx}_A(M_u) \leq \max\{\text{cx}_A(M_v), \text{cx}_A(M_w)\}$$

holds for $\{u, v, w\} = \{1, 2, 3\}$. This follows simply by comparing the k -vector space dimensions of the Tor modules in the long exact sequence

$$\cdots \rightarrow \mathrm{Tor}_n^A(M_1, k) \rightarrow \mathrm{Tor}_n^A(M_2, k) \rightarrow \mathrm{Tor}_n^A(M_3, k) \rightarrow \mathrm{Tor}_{n-1}^A(M_1, k) \rightarrow \cdots,$$

where k is the residue field of A .

In the next proposition we use the inequality (3.1) and prove that if M is an R -module with $\mathrm{cx}_S(M) \neq 1$, then $\mathrm{cx}_S(M) = \mathrm{cx}_S(\Omega_R^n(M))$ for all n . Here the assumption $\mathrm{cx}_S(M) \neq 1$ is necessary: the S -module R has a minimal free resolution

$$\cdots \rightarrow S \xrightarrow{y} S \xrightarrow{x} S \xrightarrow{y} S \xrightarrow{x} S \rightarrow 0$$

and hence has complexity one over S . However its syzygies $\Omega_R^n(R)$ are all zero for $n > 0$.

Proposition 3.2. *Let $R = S/(x)$ where S is a local ring and x is an exact zero-divisor in S . Then, for every finitely generated R -module M with $\mathrm{cx}_S(M) \neq 1$, the equality $\mathrm{cx}_S(M) = \mathrm{cx}_S(\Omega_R^n(M))$ holds for all n .*

Proof. If $\mathrm{cx}_S(M) = 0$, then Lemma 3.1 shows that $M = 0$. Thus the result is trivial in this case. Next suppose $\mathrm{cx}_S(M) > 1$. Consider the short exact sequence

$$0 \rightarrow \Omega_R^1(M) \rightarrow F \rightarrow M \rightarrow 0$$

where F is a free R -module. Since the S -module R has complexity one so does F . Hence the result follows from the inequality (3.1) and the short exact sequence considered above. \square

Next we will compare the Betti numbers and complexities of modules over R with those over S . For that we first set some notations that generalize the notion of the Betti number and the complexity of a module.

Let (A, \mathfrak{m}) be a local ring with residue field k , and M and N be A -modules with the property that $M \otimes_A N$ has finite length. Then, for every nonnegative integer n , the length of $\mathrm{Tor}_n^A(M, N)$ is finite. We define this length to be the n th Betti number $\beta_n^A(M, N)$ of the pair (M, N) , that is, $\beta_n^A(M, N) \stackrel{\mathrm{def}}{=} \ell(\mathrm{Tor}_n^A(M, N))$. The *complexity of the pair* (M, N) , denoted by $\mathrm{cx}_A(M, N)$, is then defined as:

$$\mathrm{cx}_A(M, N) \stackrel{\mathrm{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n^A(M, N) \leq an^{t-1} \text{ for all } n\}$$

Although letting $N = k$, we recover the Betti number and the complexity of M (that is, $\beta_n^A(M) = \beta_n^A(M, k)$ and $\mathrm{cx}_A(M) = \mathrm{cx}_A(M, k)$), our definition for $\mathrm{cx}_A(M, N)$ is different than the one originally defined by Avramov and Buchweitz [AvBu].

Theorem 3.3. *Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S . Furthermore, let M and N be finitely generated R -modules such that $yN = 0$ and $M \otimes_R N$ has finite length. Then, for all n ,*

$$(3.3) \quad \beta_n^R(M, N) - \sum_{i=0}^{n-2} \beta_i^R(M, N) \leq \beta_n^S(M, N) \leq \sum_{i=0}^n \beta_i^R(M, N)$$

Remark. We have used the convention that negative Betti numbers are zero.

Proof. As in the proof of Theorem 2.2, we consider the first quadrant change of rings spectral sequence:

$$\mathrm{Tor}_p^R(M, \mathrm{Tor}_q^S(R, N)) \Longrightarrow \mathrm{Tor}_{p+q}^S(M, N).$$

Since $yN = 0$, we see from Lemma 2.1 that the E^2 -page entries are given by $E_{p,q}^2 = \text{Tor}_p^R(M, N)$.

We first prove the left-hand inequality of (3.3). Fix an integer n and consider the short exact sequence

$$0 \rightarrow \Phi^{n-1}H_n \rightarrow \text{Tor}_n^S(M, N) \rightarrow E_{n,0}^\infty \rightarrow 0,$$

where $\Phi^i H_n$ is the filtration of H_n from the proof of Theorem 2.2. Since $E_{n,0}^\infty = \text{Ker } d_{n,0}^n$, we obtain the inequality $\ell(\text{Tor}_n^S(M, N)) \geq \ell(\text{Ker } d_{n,0}^n)$. Now for all $2 \leq p \leq n$, there is an exact sequence

$$0 \rightarrow \text{Ker } d_{n,0}^p \rightarrow \text{Ker } d_{n,0}^{p-1} \rightarrow \text{Im } d_{n,0}^p \rightarrow 0,$$

which implies

$$\begin{aligned} \ell(\text{Ker } d_{n,0}^n) &= \ell(\text{Ker } d_{n,0}^{n-1}) - \ell(\text{Im } d_{n,0}^n) \\ &= \ell(\text{Ker } d_{n,0}^{n-2}) - (\ell(\text{Im } d_{n,0}^{n-1}) + \ell(\text{Im } d_{n,0}^n)) \\ &\vdots \\ &= \ell(\text{Ker } d_{n,0}^1) - \sum_{i=2}^n \ell(\text{Im } d_{n,0}^i). \end{aligned}$$

For $2 \leq i \leq n$, the image of $d_{n,0}^i$ is a submodule of $E_{n-i,i-1}^i$, and the latter is a subquotient of $E_{n-i,i-1}^2$. Then since $E_{n-i,i-1}^2 = \text{Tor}_{n-i}^R(M, N)$, there is an inequality $\ell(\text{Im } d_{n,0}^i) \leq \ell(\text{Tor}_{n-i}^R(M, N))$. Moreover the module $E_{n,0}^2$ is a subquotient of $\text{Ker } d_{n,0}^1$. Thus, since $E_{n,0}^2 = \text{Tor}_n^R(M, N)$, we have $\ell(\text{Ker } d_{n,0}^1) \geq \ell(\text{Tor}_n^R(M, N))$. This gives

$$\begin{aligned} \ell(\text{Tor}_n^S(M, N)) &\geq \ell(\text{Ker } d_{n,0}^n) \\ &= \ell(\text{Ker } d_{n,0}^1) - \sum_{i=2}^n \ell(\text{Im } d_{n,0}^i) \\ &\geq \ell(\text{Tor}_n^R(M, N)) - \sum_{i=0}^{n-2} \ell(\text{Tor}_i^R(M, N)), \end{aligned}$$

proving the left-hand inequality.

For the right-hand inequality, we fix an integer n and consider the short exact sequence:

$$0 \rightarrow \Phi^{p-1}H_n \rightarrow \Phi^p H_n \rightarrow E_{p,n-p}^\infty \rightarrow 0$$

for $0 \leq p \leq n$. Counting the lengths, we obtain equalities

$$\begin{aligned} \ell(\Phi^n H_n) &= \ell(\Phi^{n-1} H_n) + \ell(E_{n,0}^\infty) \\ &= \ell(\Phi^{n-2} H_n) + \ell(E_{n-1,1}^\infty) + \ell(E_{n,0}^\infty) \\ &\vdots \\ &= \sum_{i=0}^n \ell(E_{i,n-i}^\infty). \end{aligned}$$

Each $E_{i,n-i}^\infty$ is a subquotient of $E_{i,n-i}^2$, and so since $E_{i,n-i}^2 = \text{Tor}_i^R(M, N)$, we obtain the inequality $\ell(E_{i,n-i}^\infty) \leq \ell(\text{Tor}_i^R(M, N))$. Then since $\Phi^n H_n = \text{Tor}_n^S(M, N)$, we

obtain

$$\begin{aligned} \ell(\mathrm{Tor}_n^S(M, N)) &= \ell(\Phi^n H_n) \\ &= \sum_{i=0}^n \ell(E_{i, n-i}^\infty) \\ &\leq \sum_{i=0}^n \ell(\mathrm{Tor}_i^R(M, N)), \end{aligned}$$

proving the right-hand inequality. \square

As a consequence, using the right-hand side of the inequality (3.3), we obtain an upper bound for $\mathrm{cx}_S(M, N)$ in terms of the complexity of (M, N) over R .

Corollary 3.4. *Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S . Furthermore, let M and N be finitely generated R -modules such that $yN = 0$ and $M \otimes_R N$ has finite length. Then $\mathrm{cx}_S(M, N) \leq \mathrm{cx}_R(M, N) + 1$.*

Proof. If $\mathrm{cx}_R(M, N) = \infty$, then there is nothing to prove. So suppose $\mathrm{cx}_R(M, N) = c < \infty$. Then, by the definition, there exists a real number a such that $\beta_n^R(M, N) \leq an^{c-1}$ for all n . By Theorem 3.3, the inequality

$$\beta_n^S(M, N) \leq \sum_{i=0}^n \beta_i^R(M, N) \leq \sum_{i=0}^n ai^{c-1} \leq (n+1)an^{c-1}$$

holds for all n . Therefore there is a real number b such that $\beta_n^S(M, N) \leq bn^c$ for all n . This shows that $\mathrm{cx}_S(M, N) \leq c + 1$. \square

We are unaware of an example of a pair of R -modules for which the equality holds in the left-hand side of (3.3). On the other hand, the equality may occur for the right-hand side. Indeed, when the exact zero-divisors x and y are minimal generators of the maximal ideal of S , Henriques and Şega prove [HeŞ, 1.7] that the equality

$$\sum_{n=0}^{\infty} \beta_n^S(M)t^n = \frac{1}{1-t} \sum_{n=0}^{\infty} \beta_n^R(M)t^n$$

of Poincaré series holds for every finitely generated R -module M . This gives:

$$\beta_n^S(M) = \sum_{i=0}^n \beta_i^R(M)$$

However, when x and y are arbitrary, the equality of the Poincaré series stated above may fail:

Example 3.5. Let $S = k[[x]]/(x^3)$ where k is a field. Then x^2 is an exact zero divisor in S . Set $R = S/(x^2) \cong k[[x]]/(x^2)$. It can be seen that:

$$\sum_{n=0}^{\infty} \beta_n^S(k)t^n = \frac{1}{1-t} = \sum_{n=0}^{\infty} \beta_n^R(k)t^n$$

This example also shows that the inequality of Corollary 3.4 can be strict.

We now give an example illustrating the fact that the left-hand inequality of (3.3) does give useful lower bounds in some cases:

Example 3.6. Let $R = k[x_1, \dots, x_e]/(x_1, \dots, x_e)^2$, and M be a finitely generated R -module. Then $\Omega_R^1(M)$ is a finite dimensional vector space over k of dimension $\beta_1^R(M)$. It is easy to see that the Betti numbers of k are $\beta_n^R(k) = e^n$. It follows that $\beta_n^R(M) = \beta_1^R(M)e^{n-1}$ for all $n \geq 1$. From the left-hand inequality of (3.3) we have

$$\begin{aligned} \beta_n^S(M) &\geq \beta_n^R(M) - \sum_{i=0}^{n-2} \beta_i^R(M) \\ &= \beta_1^R(M)e^{n-1} - \left(\sum_{i=1}^{n-2} \beta_1^R(M)e^{i-1} \right) - \beta_0^R(M) \\ &= \beta_1^R(M) \left(e^{n-1} - \frac{e^{n-2} - 1}{e - 1} \right) - \beta_0^R(M) \\ &= \beta_1^R(M) \left(\frac{e^n - e^{n-1} - e^{n-2} + 1}{e - 1} \right) - \beta_0^R(M) \\ &\geq \frac{\beta_1^R(M)}{2} e^{n-1} - \beta_0^R(M) \end{aligned}$$

for $e \geq 2$ and for any ring S such that there exists an exact zero-divisor x with $R \cong S/(x)$. In particular, R -modules must have exponential growth over S as well. As a specific example, let $S = k[x, y, z]/(x^2, y^2, z^2, yz)$. Then x is an exact zero-divisor in S , and $R = S/(x) \cong k[y, z]/(y, z)^2$ has the form above.

When $N = k$, the assumptions that $M \otimes N$ has finite length and $yN = 0$ hold automatically. Therefore, in this situation, Theorem 3.3 and Corollary 3.4 can be summarized as follows:

Corollary 3.7. *Let $R = S/(x)$ where S is a local ring and x is an exact zero-divisor in S . Then, for every finitely generated R -module M , the inequalities*

$$\beta_n^R(M) - \sum_{i=0}^{n-2} \beta_i^R(M) \leq \beta_n^S(M) \leq \sum_{i=0}^n \beta_i^R(M)$$

hold for all n . Consequently $\text{cx}_S(M) \leq \text{cx}_R(M) + 1$ holds.

Remark. It follows from [AHS, 4.4] that R is a complete intersection if and only if S is a complete intersection. The complexity inequality obtained in Corollary 3.7 gives a different proof for this result in case R is a complete intersection: if $\text{cx}_R(k) < \infty$, where k is the residue field of R , then it follows from Corollary 3.7 that $\text{cx}_S(k) < \infty$ and hence, by [Gu2, 2.5], S is a complete intersection.

Another observation related to the result stated above concerns commutative local Cohen-Macaulay Golod rings [Av3, 5.2]. Assume S is such a ring. Since a finitely generated module has infinite complexity over S in case it has infinite projective dimension over S and $\text{codepth}(S) \geq 2$ [Av3, 5.3.3(2)], we conclude $\text{codepth}(S) \leq 1$ (Recall $\text{cx}_S(R) = 1$). Moreover, as x is not regular, $\text{codepth}(S) = 1$. This implies that S is a hypersurface and hence R is a complete intersection.

As discussed in the introduction, when x is regular the complexity inequality is quite different than the one obtained in Corollary 3.7. More precisely, in that case the inequalities $\text{cx}_S(M) \leq \text{cx}_R(M) \leq \text{cx}_S(M) + 1$ hold. In particular the complexity of M over R is finite if and only if it is finite over S . However, in our situation, when x is an exact zero-divisor, we are unable to deduce any further inequalities,

such as $\text{cx}_R(M) \leq \text{cx}_S(M)$, from Theorem 3.3. In fact we do not know whether there exists an R -module M with $\text{cx}_S(M) < \infty$ and $\text{cx}_R(M) = \infty$. We record this in the next question.

Question. Let $R = S/(x)$ where S is a local ring and x is an exact zero-divisor in S . Is $\text{cx}_R(M) \leq \text{cx}_S(M)$ for all finitely generated R -modules M ?

4. CANONICAL ELEMENTS OF $\text{Ext}_R^2(M, M)$ AND LIFTING

In this section we restrict our attention to the case where (x, x) is a pair of exact zero-divisors in the local ring S , and $R = S/(x)$. We discuss natural chain endomorphisms of complexes over R , paralleling the construction in [Eis, Section 1], and show that whether or not they are null-homotopic dictates the liftability of R -modules to S . These results generalize classical results (for which we have no good reference) for lifting modules modulo a regular element to modulo the square of the regular element.

Canonical endomorphisms of complexes. Let

$$(1) \quad F : \cdots \rightarrow F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \rightarrow \cdots$$

be a complex of finitely generated free R -modules. We let

$$(2) \quad \tilde{F} : \cdots \rightarrow \tilde{F}_{i+1} \xrightarrow{\tilde{\partial}_{i+1}} \tilde{F}_i \xrightarrow{\tilde{\partial}_i} \tilde{F}_{i-1} \rightarrow \cdots$$

denote a preimage over S of the complex F , that is, a sequence of homomorphisms $\tilde{\partial}_i : \tilde{F}_i \rightarrow \tilde{F}_{i-1}$ of free S -modules such that F and $\tilde{F} \otimes_S R$ are isomorphic R -complexes. From the fact that $\tilde{\partial}_{i-1}\tilde{\partial}_i(\tilde{F}_i) \subseteq x\tilde{F}_{i-2}$ for all i , we can write

$$(3) \quad \tilde{\partial}_{i-1}\tilde{\partial}_i = x\tilde{s}_i$$

for some homomorphism $\tilde{s}_i : \tilde{F}_i \rightarrow \tilde{F}_{i-2}$. Now we define the homomorphisms $s_i : F_i \rightarrow F_{i-2}$ by

$$(4) \quad s_i = \tilde{s}_i \otimes_S R$$

for all i .

There are several properties of the s_i which we should like to prove.

Proposition 4.1. *The definition of s_i is independent of the factorization in (3).*

Proof. Suppose that $\tilde{\partial}_{i-1}\tilde{\partial}_i = x\tilde{t}_i$ is another factorization. Then $x(\tilde{s}_i - \tilde{t}_i) = 0$. Since $\text{ann}_S(x) = (x)$ it follows that $\tilde{s}_i - \tilde{t}_i = xu_i$ for some $u_i : \tilde{F}_i \rightarrow \tilde{F}_{i-2}$. Thus, modulo x , we have $s_i = t_i$. \square

Proposition 4.2. *The family $s = \{s_i\}$ is a chain endomorphism of F of degree -2 .*

Proof. For each i we have

$$x(\tilde{s}_{i-1}\tilde{\partial}_i - \tilde{\partial}_{i-2}\tilde{s}_i) = \tilde{\partial}_{i-2}\tilde{\partial}_{i-1}\tilde{\partial}_i - \tilde{\partial}_{i-2}\tilde{\partial}_{i-1}\tilde{\partial}_i = 0.$$

It follows that $\tilde{s}_{i-1}\tilde{\partial}_i - \tilde{\partial}_{i-2}\tilde{s}_i = xu_i$ for some $u_i : \tilde{F}_i \rightarrow \tilde{F}_{i-3}$. Hence $s_{i-1}\partial_i = \partial_{i-2}s_i$. \square

Proposition 4.3. *Let*

$$G : \cdots \rightarrow G_{i+1} \xrightarrow{\delta_{i+1}} G_i \xrightarrow{\delta_i} G_{i-1} \rightarrow \cdots$$

be another complex of finitely generated free R -modules, and assume that there exists a chain map $f : F \rightarrow G$. Let $t = \{t_i = \tilde{t}_i \otimes_S R : G_i \rightarrow G_{i-2}\}$ be the chain map defined by the factorizations $\tilde{\delta}_{i-1}\tilde{\delta}_i = x\tilde{t}_i$ for all i , where \tilde{G} is a preimage over S of G . Then the chain maps fs and tf are homotopic.

Proof. Assume that the degree of f is d . For each i let $\tilde{f}_i : \tilde{F}_i \rightarrow \tilde{G}_{i+d}$ be a preimage of $f_i : F_i \rightarrow G_{i+d}$ over S . Since $f_{i-1}\partial_i = \delta_{i+d}f_i$ for all i there exist $\tilde{h}_i : \tilde{F}_i \rightarrow \tilde{G}_{i+d-1}$ such that $\tilde{f}_{i-1}\tilde{\partial}_i - \tilde{\delta}_{i+d}\tilde{f}_i = x\tilde{h}_i$. Now for all i we have

$$\begin{aligned} x(\tilde{f}_{i-2}\tilde{s}_i - \tilde{t}_{i+d}\tilde{f}_i) &= \tilde{f}_{i-2}\tilde{\partial}_{i-1}\tilde{\partial}_i - \tilde{\delta}_{i+d-1}\tilde{\delta}_{i+d}\tilde{f}_i \\ &= (\tilde{\delta}_{i+d-1}\tilde{f}_{i-1} + x\tilde{h}_{i-1})\tilde{\partial}_i - \tilde{\delta}_{i+d-1}(\tilde{f}_{i-1}\tilde{\partial}_i - x\tilde{h}_i) \\ &= x(\tilde{h}_{i-1}\tilde{\partial}_i + \tilde{\delta}_{i+d-1}\tilde{h}_i). \end{aligned}$$

It follows that there exists for each i homomorphisms $u_i : \tilde{F}_i \rightarrow \tilde{G}_{i+d-2}$ such that

$$(\tilde{f}_{i-2}\tilde{s}_i - \tilde{t}_{i+d}\tilde{f}_i) - (\tilde{h}_{i-1}\tilde{\partial}_i + \tilde{\delta}_{i+d-1}\tilde{h}_i) = xu_i.$$

Thus for all i we have

$$f_{i-2}s_i - t_{i+d}f_i = h_{i-1}\partial_i + \delta_{i+d-1}h_i$$

where $h_i = \tilde{h}_i \otimes_S R$ for all i . This shows that fs and tf are homotopic. \square

Corollary 4.4. *The definition of the s_i is independent, up to homotopy, of the preimage \tilde{F} of F chosen in (2).*

Proof. Apply the previous proposition when f is the identity map. \square

The group $\text{Ext}_A^n(M, M)$. Let A be an associative ring, and M an A -module. Suppose that F is a projective resolution of M . Then $\text{H}^n(\text{Hom}_A(F, F))$ is the group of homotopy equivalence classes of chain endomorphisms of F of degree n . For a chain endomorphism s of F of degree n , we let $[s]$ denote the class of s in $\text{H}^n(\text{Hom}_A(F, F))$. Let G be another projective resolution of M over A . Then the comparison maps $f : F \rightarrow G$ and $g : G \rightarrow F$ lifting the identity map on M are homotopically equivalent. That is, fg is homotopic to the identity map on G and gf is homotopic to the identity map on F . It follows that the map

$$(5) \quad \theta_G^F : \text{H}^n(\text{Hom}_A(F, F)) \rightarrow \text{H}^n(\text{Hom}_A(G, G))$$

given by $[s] \mapsto [fsg]$ is an isomorphism, with inverse $\theta_F^G : [s] \mapsto [gsf]$. It is well-known that this group is $\text{Ext}_A^n(M, M)$.

Canonical elements of $\text{Ext}_R^2(M, M)$. Returning to the situation where $R = S/(x)$ for the pair (x, x) of exact zero-divisors, let F be a free resolution of M over R , and s be the endomorphism of F defined by (4). Thus we have the element $[s] \in \text{H}^2(\text{Hom}_R(F, F))$. That we call $[s]$ a canonical element of $\text{Ext}_R^2(M, M)$ is reinforced by the following lemma.

Lemma 4.5. *Let $R = S/(x)$ where S is a local ring and (x, x) is a pair of exact zero-divisors in S . Suppose that F and G are free resolutions of M over R , that s*

is the canonical endomorphism of F as defined in (4), and that t is the canonical endomorphism of G as defined in (4). Then we have

$$\theta_F^G([t]) = [s]$$

where θ_F^G is the isomorphism defined in (5).

Proof. First assume that F is a minimal free resolution of M . Then the comparison map $f : F \rightarrow G$ lifting the identity map on M can be chosen to be a split injection, with splitting $g : G \rightarrow F$, also lifting the identity map on M . In particular, we have $gf = \text{id}_F$, the identity map on F .

Denote the differential on F by ∂ , and that on G by δ . Let \tilde{F} and $\tilde{\partial}$ be a preimage over S of F , and \tilde{G} and $\tilde{\delta}$ be a preimage over S of G . We choose preimages \tilde{f} of f and \tilde{g} of g over S such that $\tilde{g}\tilde{f} = \text{id}_{\tilde{F}}$.

As $g_{i-1}\delta_i = \partial_i g_i$ for all i , there exists $u_i : \tilde{G}_i \rightarrow \tilde{F}_{i-1}$ such that $\tilde{g}_{i-1}\tilde{\delta}_i = \tilde{\partial}_i\tilde{g}_i + xu_i$ for all i . Similarly, there exists $v_i : \tilde{F}_i \rightarrow \tilde{G}_{i-1}$ such that $\tilde{\delta}_i\tilde{f}_i = \tilde{f}_{i-1}\tilde{\partial}_i + xv_i$ for all i . Thus we have

$$\begin{aligned} x(\tilde{g}_{i-2}\tilde{t}_i\tilde{f}_i - \tilde{s}_i) &= \tilde{g}_{i-2}\tilde{\delta}_{i-1}\tilde{\delta}_i\tilde{f}_i - \tilde{\partial}_{i-1}\tilde{\partial}_i \\ &= (\tilde{\partial}_{i-1}\tilde{g}_{i-1} + xu_{i-1})(\tilde{f}_{i-1}\tilde{\partial}_i + xv_i) - \tilde{\partial}_{i-1}\tilde{\partial}_i \\ &= x(\tilde{\partial}_{i-1}\tilde{g}_{i-1}v_i + u_{i-1}\tilde{f}_{i-1}\tilde{\partial}_i). \end{aligned}$$

It follows that $g_{i-2}t_i f_i - s_i = \partial_{i-1}(g_{i-1}\bar{v}_i) + (\bar{u}_{i-1}f_{i-1})\partial_i$ for all i , where $\bar{u}_i = u_i \otimes_S R$ and $\bar{v}_i = v_i \otimes_S R$. We will have shown that gtf is homotopic to s with homotopy $h_i = \bar{u}_i f_i$ once we know that $\bar{u}_i f_i = g_{i-1}\bar{v}_i$ for all i . But this is easy:

$$\begin{aligned} x(u_i\tilde{f}_i - \tilde{g}_{i-1}v_i) &= (\tilde{g}_{i-1}\tilde{\delta}_i - \tilde{\partial}_i\tilde{g}_i)\tilde{f}_i - \tilde{g}_{i-1}(\tilde{\delta}_i\tilde{f}_i - \tilde{f}_{i-1}\tilde{\partial}_i) \\ &= -\tilde{\partial}_i\tilde{g}_i\tilde{f}_i + \tilde{g}_{i-1}\tilde{f}_{i-1}\tilde{\partial}_i \\ &= 0, \end{aligned}$$

hence the claim.

Notice that we also have $\theta_G^F([s]) = [t]$, when F is minimal. Therefore, for two arbitrary free resolutions F and G of M , that $\theta_F^G([t]) = [s]$ follows from composing $\theta_F^G = \theta_L^G \theta_F^L$ where L is a minimal free resolution of M . \square

Lifting. Let B be an associative ring, I an ideal of B , and $A = B/I$. Recall that a finitely generated A -module M is said to *lift* to B , with *lifting* M' , if there exists a finitely generated B -module M' such that $M \cong M' \otimes_B A$, and $\text{Tor}_i^B(M', A) = 0$ for all $i \geq 1$. Similarly, a complex of finitely generated free A -modules

$$F : \cdots \rightarrow F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \rightarrow \cdots$$

is said to *lift* to B , with *lifting* \tilde{F} , if there exists a preimage \tilde{F} of F

$$\tilde{F} : \cdots \rightarrow \tilde{F}_{i+1} \xrightarrow{\tilde{\partial}_{i+1}} \tilde{F}_i \xrightarrow{\tilde{\partial}_i} \tilde{F}_{i-1} \rightarrow \cdots$$

such that $\tilde{\partial}_{i-1}\tilde{\partial}_i = 0$ for all i . A close connection between these two notions of lifting will be explained in the next theorem. We want also to show that when $R = S/(x)$ for (x, x) a pair of exact zero-divisors, the triviality of the canonical element $[s]$ determines whether the module M lifts to S .

Theorem 4.6. *Let $R = S/(x)$ where S is a local ring and (x, x) is a pair of exact zero-divisors in S . Then for every finitely generated R -module M , the following are equivalent.*

- (1) M lifts to S .
- (2) The canonical element $[s]$ in $\text{Ext}_R^2(M, M)$ is trivial.
- (3) Every free resolution of M by finitely generated free R -modules lifts to S .
- (4) Some free resolution of M by finitely generated free R -modules lifts to S .

Proof. (1) \implies (2). Suppose that M' is a lifting of M to S . Let

$$\tilde{F} : \dots \rightarrow \tilde{F}_2 \xrightarrow{\tilde{\partial}_2} \tilde{F}_1 \xrightarrow{\tilde{\partial}_1} \tilde{F}_0 \rightarrow 0$$

be a resolution of M' by finitely generated free S -modules. Since $\text{Tor}_i^S(M', R) = 0$ for all $i > 0$, $F = \tilde{F} \otimes_S R$ is a resolution of $M \cong M' \otimes_S R$ by finitely generated free R -modules. Computing the endomorphism s from the preimage \tilde{F} of F , which is exact, we see that s is actually the zero endomorphism, and is therefore certainly trivial in $\text{Ext}_R^2(M, M)$.

(2) \implies (3). By Lemma 4.5 the canonical element of $\text{Ext}_R^2(M, M)$ is trivial regardless of which resolution by finitely generated free R -modules F of M we choose to define it. Therefore let F be an arbitrary such resolution of M , and let s be the canonical chain endomorphism defined as in (4). By assumption s is homotopic to zero. Therefore there exists a homotopy $h = \{h_i\}$ with $h_i : F_i \rightarrow F_{i-1}$ such that $s_i = \partial_{i-1}h_i + h_{i-1}\partial_i$ for all i . Let \tilde{F} be an arbitrary preimage of F , with maps $\tilde{\partial}$. Let $\tilde{h}_i : \tilde{F}_i \rightarrow \tilde{F}_{i-1}$ be a preimage of h_i for all i . There exists $u_i : \tilde{F}_i \rightarrow \tilde{F}_{i-2}$ such that $\tilde{s}_i = \tilde{\partial}_{i-1}\tilde{h}_i + \tilde{h}_{i-1}\tilde{\partial}_i + xu_i$ for all i . Now consider the preimage F^\sharp of F where we take $F_i^\sharp = \tilde{F}_i$ for all i , but we take the maps $\partial_i^\sharp = \tilde{\partial}_i - x\tilde{h}_i$ instead. We have

$$\begin{aligned} \partial_{i-1}^\sharp \partial_i^\sharp &= (\tilde{\partial}_{i-1} - x\tilde{h}_{i-1})(\tilde{\partial}_i - x\tilde{h}_i) \\ &= \tilde{\partial}_{i-1}\tilde{\partial}_i - x(\tilde{\partial}_{i-1}\tilde{h}_i + \tilde{h}_{i-1}\tilde{\partial}_i) \\ &= x(\tilde{s}_i - \tilde{\partial}_{i-1}\tilde{h}_i - \tilde{h}_{i-1}\tilde{\partial}_i) \\ &= 0. \end{aligned}$$

Thus F^\sharp is a lifting of F to S .

(3) \implies (4) is trivial. To show that (4) \implies (1), assume that F is a free resolution of M by finitely generated free R -modules, which lifts to the complex \tilde{F} over S . We claim that $H_i(\tilde{F}) = 0$ for $i \neq 0$. Indeed, if $\tilde{\partial}_i(a) = 0$, some $a \in \tilde{F}_i$, then $a = \tilde{\partial}_{i+1}(b) + xc$ for some $b \in \tilde{F}_{i+1}$ and $c \in \tilde{F}_i$ by exactness of F . Since $x\tilde{\partial}_i(c) = 0$, we have $\tilde{\partial}_i(c) \in x\tilde{F}_{i-1}$. Again by exactness of F we have $c = \tilde{\partial}_{i+1}(d) + xe$ for some $d \in \tilde{F}_{i+1}$ and $e \in \tilde{F}_i$. Therefore $a = \tilde{\partial}_{i+1}(b + xd)$. It follows that \tilde{F} is a resolution of $M' = H_0(\tilde{F})$ by finitely generated free S -modules, and thus M' is a lifting of M to S . \square

We end with an example showing that there are local rings S admitting a pair of exact zero-divisors (x, x) , but no local ring T with regular element \tilde{x} such that $S = T/(\tilde{x}^2)$ and $x = \tilde{x} + (\tilde{x}^2)$. Therefore the notion of lifting modulo an exact zero-divisor is a more general notion than lifting from modulo a regular element to modulo the square of the regular element.

Example 4.7. Let k be field, $S = k[V, X, Y, Z]/I$ where I is the ideal

$$(V^2, Z^2, XY, VX + XZ, VY + YZ, VX + Y^2, VY - X^2),$$

and set $v = V + I$. Then (v, v) is a pair of exact zero-divisors. Moreover, it can be shown from the results in [B] that S does not have an embedded deformation. Therefore there is no local ring T and non-zero-divisor \tilde{V} of T such that $S \cong T/(\tilde{V}^2)$.

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