HOCHSCHILD HOMOLOGY AND GLOBAL DIMENSION

PETTER ANDREAS BERGH & DAG MADSEN

ABSTRACT. We prove that for certain classes of graded algebras (Koszul, local, cellular), infinite global dimension implies that Hochschild homology does not vanish in high degrees, provided the characteristic of the ground field is zero. Our proof uses Igusa's formula relating the Euler characteristic of relative cyclic homology to the graded Cartan determinant.

1. INTRODUCTION

The homological properties of a finite dimensional algebra are closely related to the behavior of the algebra as a bimodule over itself. For example, if the algebra has finite projective dimension as a bimodule, then its global dimension is also finite. The converse holds if the algebra modulo its Jacobson radical is separable over the ground field, something which automatically happens when the field is algebraically closed.

In particular, if a finite dimensional algebra over an algebraically closed field has finite global dimension, then all its higher Hochschild cohomology groups vanish. In [Hap], following this easy observation, Happel remarked that "the converse seems to be not known", thus giving birth to what subsequently became known as "Happel's question": if all the higher Hochschild cohomology groups of a finite dimensional algebra vanish, then is the algebra of finite global dimension? As shown in [AvI], the answer is yes when the algebra is commutative. However, it was shown in [BGMS] that the answer in general is no. Namely, given a field k and a nonzero element $q \in k$ which is not a root of unity, then the total Hochschild cohomology of the four-dimensional algebra

$$k\langle X, Y \rangle / (X^2, XY - qYX, Y^2)$$

is five-dimensional. In particular, all the higher Hochschild cohomology groups of this algebra vanish, whereas the algebra, being selfinjective, clearly does not have finite global dimension.

As shown by Han in [Han], the total Hochschild homology of the above algebra is infinite dimensional. Han then conjectured that the homology version of Happel's question would always hold, namely that a finite dimensional algebra whose higher Hochschild homology groups vanish must be of finite global dimension. In the same paper, he showed that the conjecture

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holds for monomial algebras. Moreover, as in the cohomology case, the conjecture holds if the algebra is commutative, by [AV-P] (for finitely generated but not necessarily finite dimensional algebras, see also [V-P]).

In this paper, we show that Han's conjecture holds for graded local algebras, Koszul algebras and graded cellular algebras, provided the characteristic of the ground field is zero. We do this by exploiting some particular properties of the graded Cartan matrix and the logarithm of its determinant, concepts extensively studied in [Igu].

2. Hochschild homology and cyclic homology

Throughout this paper, we fix a field k, not necessarily algebraically closed. Let A be a finite dimensional k-algebra, and denote by A^{e} the enveloping algebra $A \otimes_k A^{\mathrm{op}}$ of A. The *Hochschild homology* of A, denoted $\mathrm{HH}_*(A)$, is defined by $\mathrm{HH}_*(A) = \mathrm{Tor}_*^{A^{\mathrm{e}}}(A, A)$. By definition, it is obtained by taking any projective bimodule resolution of A, applying $A \otimes_{A^{\mathrm{e}}} -$ and computing the homology of the resulting complex. However, we shall explore one particular such resolution, which eventually leads to the definition of cyclic homology.

For each $n \geq 0$, denote by $A^{\otimes n}$ the n-fold tensor product $A \otimes_k \cdots \otimes_k A$, in which there are n copies of A. For $n \geq 2$, this is a projective bimodule. Define a map

$$\begin{array}{rcl} A^{\otimes(n+1)} & \xrightarrow{b'} & A^{\otimes n} \\ a_0 \otimes \cdots \otimes a_n & \mapsto & \displaystyle\sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \end{array}$$

and consider the complex

$$\cdots \longrightarrow A^{\otimes 4} \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2}$$

in which $A^{\otimes 2}$ is in degree zero. By [CaE, §IX.6] this complex is exact, and it is therefore a projective bimodule resolution of A. This is the *standard resolution* (or *Bar resolution*) of A. Applying $A \otimes_{A^{\mathrm{e}}} -$ to this resolution, we obtain the complex

$$\cdots \longrightarrow A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A,$$

in which the map $A^{\otimes(n+1)} \xrightarrow{b} A^{\otimes n}$ is given by

$$\begin{array}{rcl} a_0 \otimes \cdots \otimes a_n & \mapsto & \displaystyle\sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & & +(-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{array}$$

By definition, the homology of this complex is the Hochschild homology of our algebra A.

The standard resolution and the Hochschild complex are also the key ingredients in the definition of cyclic homology. For each $n \geq 0$, define the map

$$\begin{array}{rcl} A^{\otimes(n+1)} & \xrightarrow{t} & A^{\otimes(n+1)} \\ a_0 \otimes \cdots \otimes a_n & \mapsto & (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}, \end{array}$$

and let $N = 1 + t + \cdots + t^n$ be the corresponding norm operator. Then $(1-t)b' = b(1-t)$ and $b'N = Nb$ (cf. [Lod, Lemma 2.1.1]), and so

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & \cdots
\end{array}$$

is a first quadrant double complex (in which the lower left A has degree $(0,0)$). The *cyclic homology* of A, denoted $\mathrm{HC}_*(A)$, is the homology of the resulting total complex. It is closely linked to the Hochschild homology of A via the well known long exact sequence

$$\cdots \to \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A) \xrightarrow{I} \cdots$$

due to Connes, the *SBI sequence* or *Connes' exact sequence.*

Both Hochschild and cyclic homology are functorial, since an algebra homomorphism induces a map between the corresponding Hochschild complexes and cyclic double complexes. In particular, if $\mathfrak{a}$ is a twosided ideal of A, then the surjection $A \to A/\mathfrak{a}$ induces a surjective map of cyclic double complexes. The kernel of this map is a double complex, and the homology of its total complex, denoted $\mathrm{HC}_*(A, \mathfrak{a})$, is the *relative cyclic homology* of A with respect to $\mathfrak{a}$.

Suppose the algebra A is graded, say $A = A_0 \oplus \cdots \oplus A_s$. Then its grading induces an internal grading

$$\begin{aligned}
\mathrm{HH}_*(A) &= \oplus_i \mathrm{HH}^i_*(A) \\
\mathrm{HC}_*(A) &= \oplus_i \mathrm{HC}^i_*(A) \\
\mathrm{HC}_*(A, \mathfrak{a}) &= \oplus_i \mathrm{HC}^i_*(A, \mathfrak{a})
\end{aligned}$$

on both Hochschild and (relative) cyclic homology. Consider the SBI sequence relating the Hochschild and cyclic homology of A. By a theorem of Goodwillie (cf. [Goo, Corollary II.4.6] or [Wei, Theorem 9.9.1]), the image of the map $\mathrm{HC}^i_n(A) \xrightarrow{S} \mathrm{HC}^i_{n-2}(A)$ is annihilated by i (where i is viewed as an element in k) for every $i > 0$. In particular, if the characteristic of k is zero, then we obtain a short exact sequence

$$(\dagger) \qquad 0 \to \mathrm{HC}^i_{n-1}(A) \to \mathrm{HH}^i_n(A) \to \mathrm{HC}^i_n(A) \to 0$$

for each $i > 0$. It follows from this sequence that if $\mathrm{HC}^i_n(A)$ is nonzero for some $i > 0$, then so is $\mathrm{HH}^i_{n+1}(A)$. Next, suppose A_0 is a product of copies of the ground field k, say $A_0 = k^{\times r}$, and let J denote the radical $A_1 \oplus \cdots \oplus A_s$ of A. If the characteristic of k is zero, then it follows from [Igu, Corollary 1.2] that $\mathrm{HC}^m_n(A, J) = 0$ for any $m \geq 1$ and $n \geq m$. Therefore, in this case, the *Euler characteristic* of $\mathrm{HC}_*^m(A,J)$ is well defined and given by

$$\chi\left(\mathrm{HC}_*^m(A,J)\right) = \sum_{n=0}^{m-1}(-1)^i \dim_k \mathrm{HC}_n^m(A,J),$$

and we define the *graded Euler characteristic* by

$$\chi(\mathrm{HC}_*(A,J))(x) \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \chi(\mathrm{HC}_*^m(A,J))x^m.$$

The latter is a power series with integer coefficients.

The aim of this paper is to establish results concerning the non-vanishing of Hochschild homology for certain algebras. To simplify the notation, we therefore define the following:

$$\begin{aligned}
\operatorname{hhdim} A &\stackrel{\text{def}}{=} \sup\{n \in \mathbb{Z} \mid \mathrm{HH}_n(A) \neq 0\} \\
\operatorname{chdim} A &\stackrel{\text{def}}{=} \sup\{n \in \mathbb{Z} \mid \mathrm{HC}_n(A) \neq 0\} \\
\operatorname{chdim}(A,\mathfrak{a}) &\stackrel{\text{def}}{=} \sup\{n \in \mathbb{Z} \mid \mathrm{HC}_n(A,\mathfrak{a}) \neq 0\}.
\end{aligned}$$

As mentioned in the introduction, we show in this paper that if the characteristic of k is zero and A has infinite global dimension, then $\operatorname{hhdim} A = \infty$ when A is graded local, Koszul or graded cellular. We end this section with the following result, which shows that we only need to establish the non-vanishing of the relative cyclic homology of A with respect to the radical.

Lemma 2.1. *Suppose A is graded and that its degree zero part is a product of copies of k. Furthermore, suppose the characteristic of k is zero. Then*

$$\operatorname{chdim}(A,J) = \infty \;\Leftrightarrow\; \operatorname{hhdim} A = \infty,$$

where J is the radical of A.

Proof. By construction, there is a long exact sequence

$$\cdots \to \mathrm{HC}_{n+1}(A_0) \to \mathrm{HC}_n(A,J) \to \mathrm{HC}_n(A) \to \mathrm{HC}_n(A_0) \to \cdots$$

relating relative and ordinary cyclic homology. Since A_0 lives only in degree zero, its internal positive degree cyclic homology vanishes, i.e. $\mathrm{HC}_*^m(A_0) = 0$ for $m > 0$. Consequently, there is an isomorphism $\mathrm{HC}_n^m(A,J) \simeq \mathrm{HC}_n^m(A)$ for every n and any $m > 0$. Suppose $\mathrm{HC}_n(A,J)$ is nonzero for some n. Since A_0 is a product of copies of k, we know from above that $\mathrm{HC}_n^i(A,J) = 0$ for $i \leq n$, and therefore there must be an integer $m > n$ such that $\mathrm{HC}_n^m(A,J)$ is nonzero. The isomorphism above then shows that $\mathrm{HC}_n^m(A)$ is nonzero, and from the exact sequence (†) we see that the same holds for $\mathrm{HH}_n^m(A)$. This shows the implication $\Rightarrow$. For the reverse implication, note that for $n > 0$, the group $\mathrm{HH}_n^0(A)$ vanishes. Hence if $\mathrm{HH}_n(A)$ is nonzero, then from the SBI sequence we see that there is an $m > 0$ such that either $\mathrm{HC}_n^m(A)$ or $\mathrm{HC}_{n-1}^m(A)$ is nonzero. Then either $\mathrm{HC}_n^m(A,J)$ or $\mathrm{HC}_{n-1}^m(A,J)$ must be nonzero. □

3. The graded Cartan determinant

In this section A denotes a positively graded finite dimensional k-algebra $A = A_0 \oplus A_1 \oplus \cdots \oplus A_s$. We assume $A_0 \simeq k \times \cdots \times k = k^{\times r}$ as rings. Let $1_A = e_1 + \ldots + e_r$ be the corresponding decomposition of the identity.

In [Igu], the author presented a formula relating the graded Euler characteristic $\chi(\mathrm{HC}_*(A, J))(x)$ of relative cyclic homology to the so-called graded Cartan determinant of A. For $0 \leq l \leq s$, let C^l be the $r \times r$ matrix with entries $C^l_{i,j} = \dim_k e_j A_l e_i$. The *graded Cartan matrix* of A is defined to be the $r \times r$ matrix

$$C_A(x) = C^0 + C^1 x + C^2 x^2 + \ldots + C^s x^s$$

with entries in $\mathbb{Z}[x]$. Its determinant $\det C_A(x)$ is the *graded Cartan determinant* of A. With our assumptions C^0 is the identity matrix, and therefore $\det C_A(x)$ is a polynomial of degree $u \leq s$ with integer coefficients and constant term 1. The logarithm of the determinant is then a power series that can be defined according to the formula

$$\log \det C_A(x) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(\det C_A(x) - 1)^m}{m}.$$

Although this power series in general has rational coefficients, its formal derivative $D_x(\log \det C_A(x))$ has integer coefficients. We say that a power series is *proper* if it has infinitely many nonzero terms.

Lemma 3.1. *If $\det C_A(x) \neq 1$, then the power series $D_x(\log \det C_A(x))$ is proper and has integer coefficients $\{b_i\}_{i \geq 0}$. Moreover, the sequence $\{b_i\}_{i \geq 0}$ satisfies a linear recurrence relation of order u with constant integer coefficients.*

Proof. The chain rule gives

$$D_x(\log \det C_A(x)) \cdot \det C_A(x) = D_x(\det C_A(x)),$$

or alternatively

$$D_x(\log \det C_A(x)) = (\det C_A(x))^{-1} \cdot D_x(\det C_A(x)).$$

Since the degree of the polynomial $D_x(\det C_A(x))$ is strictly less than the degree of $\det C_A(x)$, it follows from the first formula that $D_x(\log \det C_A(x))$ must be a proper power series. Since $D_x(\det C_A(x))$ has integer coefficients, it follows from the second formula that $D_x(\log \det C_A(x))$ has integer coefficients.

If $\det C_A(x) = 1 + c_1 x + \ldots + c_u x^u$, then for any $m \geq u$, the first formula gives $b_m + c_1 b_{m-1} + \ldots + c_u b_{m-u} = 0$. The last statement of the lemma follows. □

Next we state Igusa's formula as presented in [Igu]. Recall that the Möbius function μ is the multiplicative number theoretic function defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^t & \text{if } n \text{ is a product of } t \text{ distinct primes,} \\ 0 & \text{if } n \text{ has one or more repeated prime factors.} \end{cases}$$

Theorem 3.2. [Igu, Theorem 3.5] *Let A be a graded algebra over a field k of characteristic zero, and suppose A_0 is a product of copies of k. Then*

(a)
$$\chi(\mathrm{HC}_*(A,J))(x) = \sum_{m=1}^{\infty} \log\det C_A(x^m) \sum_{d|m} \frac{\mu(d)}{d},$$
(b)
$$\log\det C_A(x) = \sum_{m=1}^{\infty} \chi(\mathrm{HC}_*(A,J))(x^m) \sum_{d|m} \frac{d\mu(d)}{m},$$
where μ is the Möbius function.

An immediate corollary of this theorem is that $\det C_A(x) = 1$ if and only if $\chi(\mathrm{HC}_*(A,J))(x) = 0$. Our aim is to show that if $\det C_A(x) \neq 1$, then $\chi(\mathrm{HC}_*(A,J))(x)$ is a proper power series.

We would like to have a formula relating the coefficients of
$$D_x(\log\det C_A(x)) = \sum_{i=0}^{\infty} b_i x^i$$
with the coefficients of
$$\chi(\mathrm{HC}_*(A,J))(x) = \sum_{i=1}^{\infty} a_i x^i.$$
For this purpose we introduce the number theoretic function θ, defined as follows.
$$\theta(m) = \sum_{d|m} d\mu(d) = \prod_{\substack{p|m \\ p \text{ prime}}} (1-p).$$
The function θ is multiplicative and has the property that if $n \mid m$, then $\theta(n) \mid \theta(m)$. From Theorem 3.2 (b) we get
$$D_x(\log\det C_A(x)) = \sum_{m=1}^{\infty} [\sum_{d|m} a_d \cdot d \cdot \theta(\frac{m}{d})] x^{m-1}.$$
For convenience let
$$f(m) = \sum_{d|m} a_d \cdot d \cdot \theta(\frac{m}{d}).$$
With this notation $f(m) = b_{m-1}$.

If $\chi(\mathrm{HC}_*(A,J))(x)$ has only finitely many non-zero coefficients, in other words if $\chi(\mathrm{HC}_*(A,J))(x)$ is a polynomial, then we get the following condition on the function f.

Proposition 3.3. *If $\chi(\mathrm{HC}_*(A,J))(x)$ is a polynomial, then for every pair of integers $s, t > 0$ there are s consecutive positive integers $N+1, \ldots, N+s$ such that $2^t \mid f(N+i)$, $1 \leq i \leq s$.*

Proof. Let v be the degree of $\chi(\mathrm{HC}_*(A,J))(x)$. Choose st different odd prime numbers $p_{i,j} > v$, $1 \leq i \leq s, 1 \leq j \leq t$. Let $n_i = \prod_{j=1}^{t} p_{i,j}$ for each $1 \leq i \leq s$. The system of congruences
$$x \equiv -1 \mod n_1,$$
$$x \equiv -2 \mod n_2,$$
$$\vdots$$

$$x \equiv -s \mod n_s$$

has a unique solution modulo $n_1 n_2 \cdots n_s$. Let $N > 0$ be a solution. Now for each $1 \leq i \leq s$, we have $n_i \mid N+i$. Also $n_i \mid \frac{N+i}{d}$ whenever $d \leq v$ and $d \mid N+i$. Since $a_d = 0$ for $d > v$, we have

$$f(N+i) = \sum_{\substack{d|(N+i) \\ d \leq v}} a_d \cdot d \cdot \theta(\frac{N+i}{d}).$$

Since $\theta(n_i) \mid \theta(\frac{N+i}{d})$ whenever $d \leq v$ and $d \mid N+i$, it follows that $\theta(n_i) \mid f(N+i)$. Since $\theta(n_i) = \prod_{j=1}^{t}(1-p_{i,j})$, it follows that $2^t \mid \theta(n_i) \mid f(N+i)$. □

If $\det C_A(x) \neq 1$, then f cannot satisfy this condition, and as a consequence we get our theorem.

Theorem 3.4. *Let A be a graded finite dimensional algebra over a field k of characteristic zero, and suppose A_0 is a product of copies of k. If $\det C_A(x) \neq 1$, then $\chi(\mathrm{HC}_*(A, J))(x)$ is a proper power series.*

Proof. Suppose for contradiction that $\det C_A(x) \neq 1$ and $\chi(\mathrm{HC}_*(A, J))(x)$ is a polynomial. Since $\det C_A(x) \neq 1$, it follows that $\chi(\mathrm{HC}_*(A, J))(x) \neq 0$. Let v be the degree of $\chi(\mathrm{HC}_*(A, J))(x)$, and let l be the degree of its lowest degree nonzero term. Note that $l > 0$ since $\chi(\mathrm{HC}_*(A, J))(x)$ has zero constant term. Let $p > v$ be a prime number. Then $f(lp^i) = a_l \cdot l \cdot \theta(p^i) = a_l \cdot l(1-p) = f(lp) \neq 0$ for any $i \geq 1$. Let $t \geq 1$ be the number such that $2^{t-1} \mid f(lp)$ but $2^t \nmid f(lp)$.

Let $u > 0$ be the degree of $\det C_A(x)$. By Proposition 3.3 there are u consecutive positive integers $N+1, \dots, N+u$ such that $2^t \mid f(N+i)$, $1 \leq i \leq u$. So $D_x(\log \det C_A(x))$ has u consecutive coefficients $b_N, \dots, b_{N+u-1}$ which are divisible by 2^t. But then by Lemma 3.1 we have $2^t \mid b_m$ for any $m \geq N$, so $2^t \mid f(m)$ for any $m \geq N+1$. This contradicts the fact that $2^t \nmid f(lp^i)$ for any $i \geq 0$. □

For our investigations into the validity of Han's conjecture, the following corollary is important.

Corollary 3.5. *Let A be a graded finite dimensional algebra over a field k of characteristic zero, and suppose A_0 is a product of copies of k. Suppose $\det C_A(x) \neq 1$. Then*

(a) $\mathrm{chdim}(A, J) = \infty$,
(b) $\mathrm{hhdim}\, A = \infty$.

Proof. Parts (a) and (b) are equivalent by Lemma 2.1. Since A is finite dimensional, for each $n \geq 0$ the group $\mathrm{HC}_n(A, J)$ is finite dimensional and therefore $\mathrm{HC}_n^m(A, J) \neq 0$ only for finitely many internal degrees m. If $\chi(\mathrm{HC}_*(A, J))(x)$ is a proper power series, then $\mathrm{HC}_*^m(A, J) \neq 0$ for infinitely many internal degrees m, and this is only possible if $\mathrm{chdim}(A, J) = \infty$. So it follows from Theorem 3.4 that if $\det C_A(x) \neq 1$, then $\mathrm{chdim}(A, J) = \infty$. □

4. Non-vanishing of Hochschild homology in high degrees

In this section we apply Corollary 3.5 to various classes of graded algebras (Koszul, local, cellular), and prove that Han's conjecture [Han] holds for these classes. Han's conjecture can be stated as follows.

Conjecture (Han)**.** Let A be a finite dimensional algebra over a field. If A has infinite global dimension, then $\operatorname{hhdim} A = \infty$.

Let A be a graded finite dimensional k-algebra, and suppose $A_0 \simeq k^{\times r}$. The matrix $C_A(x)$ is invertible as a matrix over $\mathbb{Z}[x]$ if and only if $\det C_A(x) = 1$. On the other hand, since $\det C_A(x)$ has constant term 1, the matrix $C_A(x)$ is always invertible when considered as a matrix over $\mathbb{Z}[[x]]$. In [Wil] we find the following formula for the entries in the inverse matrix $C_A^{-1}(x)$. Here $\operatorname{Ext}_{\operatorname{gr} A}$ denotes graded extensions and $X[u]$ denotes the uth shift of the graded module X. For each $1 \leq i \leq r$, we let S_i denote the degree zero simple module $S_i = Ae_i/JAe_i$.

Theorem 4.1. [Wil, Theorem 1.5] *The entry c_{ij} in $C_A^{-1}(x)$ is given by*

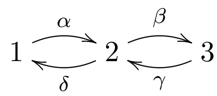
$$c_{ij} = \sum_{u \geq 0} \sum_{v \geq 0} (-1)^v \dim_k \left(\operatorname{Ext}^v_{\operatorname{gr} A}(S_i, S_j[u]) \right) x^u.$$

For $C_A(x)$ to be invertible as a matrix over $\mathbb{Z}[x]$, all entries c_{ij} above have to be polynomials (not proper power series), and we get the following corollary.

Corollary 4.2. $\det C_A(x) = 1$ *if and only if the entries c_{ij} in $C_A^{-1}(x)$ are polynomials for all $1 \leq i, j \leq r$.*

If A has finite global dimension, then $\det C_A(x) = 1$. The converse is not true, there are algebras A of infinite global dimension with $\det C_A(x) = 1$.

Example 4.3. Let A be the path algebra $A = kQ/I$, where Q is the quiver



and $I = \langle \rho \rangle$ is the ideal generated by the set of relations $\rho = \{\beta\alpha, \gamma\beta, \beta\gamma, \delta\gamma, \alpha\delta\alpha, \delta\alpha\delta\}$. Then $\det C_A(x) = 1$, but A has infinite global dimension. Since A is monomial, Han's conjecture is known to hold for this algebra and therefore $\operatorname{hhdim} A = \infty$.

For some classes of graded algebras, the combination $\det C_A(x) = 1$ and infinite global dimension is not possible. We can use Corollary 3.5 to prove that Han's conjecture holds for these classes.

4.1. **Koszul algebras.** Let A be a graded algebra with $A_0 \simeq k^{\times r}$. Such an algebra is called *Koszul* ([Pri], [BGS]) if $\operatorname{Ext}^v_{\operatorname{gr} A}(S_i, S_j[u]) \neq 0$ implies $u = v$. More generally, let $N \geq 2$ be an integer, and define a function

$$n \colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$$

by $n(2q) = Nq$ and $n(2q+1) = Nq+1$. Then the algebra A is said to be N-*Koszul* ([Ber, Proposition 2.14]) if $\operatorname{Ext}^v_{\operatorname{gr} A}(S_i, S_j[u]) \neq 0$ implies $u = n(v)$. The notion of N-Koszul algebras was introduced by Berger in [Ber], and generalizes that of ordinary Koszul algebras, which is the case $N = 2$.

Theorem 4.4. *Let A be a finite dimensional N-Koszul algebra over a field of characteristic zero. If A has infinite global dimension, then* $\operatorname{hhdim} A = \infty$.

Proof. Suppose $\det C_A(x) = 1$. Then $C_A(x)$ has an inverse $C_A^{-1}(x)$ whose entries are polynomials and *not* proper power series. Since our algebra is N-Koszul, we know that $\operatorname{Ext}^v_{\operatorname{gr} A}(S_i, S_j[u])$ is nonzero only when $u = n(v)$, and so we may simplify the formula for the entries in the inverse matrix and get

$$c_{ij} = \sum_{v \geq 0} (-1)^v \dim_k \left(\operatorname{Ext}^v_{\operatorname{gr} A}(S_i, S_j[n(v)]) \right) x^{n(v)}.$$

By assumption, all the entries in $C_A^{-1}(x)$ are polynomials, hence all the higher graded extension groups between simple A-modules vanish. Consequently, the global dimension of A is finite. This shows that the determinant of the graded Cartan matrix of an N-Koszul algebra of infinite global dimension is not one, and so by Corollary 3.5 we are done. □

4.2. **Local algebras.** If A is graded finite dimensional with $A_0 = k$, then the graded Cartan determinant is by definition the same as the Hilbert polynomial.

Theorem 4.5. *Suppose k is of characteristic zero, and let A be a graded finite dimensional k-algebra with $A_0 = k$. If A has infinite global dimension, then* $\operatorname{hhdim} A = \infty$.

Proof. If $\operatorname{hhdim} A < \infty$, then $\det C_A(x) = 1$ by Corollary 3.5. If the Hilbert polynomial of A is 1, then $A = k$ and A has finite global dimension. □

This theorem can be seen as a special case of the following more general result.

Theorem 4.6. *Suppose k is of characteristic zero, let $Q = (Q_0, Q_1)$ be a finite oriented quiver, and let $\mathtt{J}$ be the ideal in the path algebra kQ generated by the arrows. Furthermore, let I be a homogeneous ideal in kQ such that $\mathtt{J}^t \subseteq I \subseteq \mathtt{J}^2$ for some t. If Q contains a loop, then* $\operatorname{hhdim} kQ/I = \infty$.

Proof. This follows from Corollary 3.5 and the construction of the graded Cartan matrix. The entries in $C_{kQ/I}(x)$ which are not on the diagonal have constant term zero, so the only contribution to the degree one coefficient in the determinant comes from the product of the diagonal entries. The ith diagonal entry is of the form $1 + n_i x + \cdots$, where n_i is the number of loops at vertex i of Q. Therefore

$$\det C_{kQ/I}(x) = 1 + \sum_{i=1}^{|Q_0|} n_i x + \text{higher terms}.$$

Consequently, if Q contains a loop, then $\det C_{kQ/I}(x) \neq 1$. □

4.3. **Cellular algebras.** Cellular algebras were introduced in [GL] (see also [KX1]) and are finite dimensional algebras which admit a special kind of basis. A k-algebra A is called a *cellular algebra* with cell datum (Λ, M, C, i) if all of the following three conditions are satisfied.

(C1) The set Λ is finite and partially ordered. Associated with each $\lambda \in \Lambda$ there is a finite set $M(\lambda)$. The algebra A has a k-basis $\{C^{\lambda}_{S,T}\}$, where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in \Lambda$.

(C2) The map i is a k-linear anti-automorphism of A with $i^2 = 1_A$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ and each $a \in A$, the product $a \cdot C^{\lambda}_{S,T}$ can be written as

$$a \cdot C^{\lambda}_{S,T} = \sum_{U \in M(\lambda)} h_a(U,S) \cdot C^{\lambda}_{U,T} + h',$$

where h' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $h_a(U,S)$ do not depend on T.

Let $S_1, \dots, S_r$ be a complete set of non-isomorphic simple A-modules, and let $P_1, \dots, P_r$ denote the corresponding indecomposable projective modules. The *(ungraded) Cartan matrix* of A, denoted U_A, is the $r \times r$ matrix over $\mathbb{Z}$ where for all $1 \leq i, j \leq r$, the entry $(U_A)_{ij}$ is equal to the composition multiplicity of S_j in P_i. In [KX2] we find the following characterization of cellular algebras of finite global dimension. For the definition of quasi-hereditary algebras, see [CPS].

Theorem 4.7. [KX2, Theorem 1.1] *Let A be a cellular algebra over a field. The following are equivalent.*

(a) *A is quasi-hereditary,*
(b) *A has finite global dimension,*
(c) $\det U_A = 1$.

If A is graded with $A_0 \simeq k^{\times r}$, then U_A can be obtained from $C_A(x)$ by evaluating for $x = 1$. Therefore $\det U_A = \det C_A(1)$.

Theorem 4.8. *Suppose k is of characteristic zero, and let A be a graded cellular k-algebra such that A_0 is a product of copies of k. If A has infinite global dimension, then* $\operatorname{hhdim} A = \infty$.

Proof. If $\det U_A \neq 1$, then $\det C_A(x) \neq 1$, and so by Corollary 3.5 we have $\operatorname{hhdim} A = \infty$. □

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Institutt for matematiske fag, NTNU, 7491 Trondheim, Norway
E-mail address: bergh@math.ntnu.no
E-mail address: dagma@math.ntnu.no