Solution

Ex. 3.3 a

a given, fixed design matrix

Gauss-Markov: Model is $\underset{N\times 1}{\underset{\sim}{Y}} = \underset{N\times p+1}{\underset{\sim}{X}}\, \underset{p+1\times 1}{\underset{\sim}{\beta}} + \underset{N\times 1}{\underset{\sim}{\varepsilon}}$

Wants to estimate $a^T\beta$ with unbiased estimator

LS: Use $a^T\hat{\beta}$. Then $E(a^T\hat{\beta}) = a^T\beta$
since $\hat{\beta}$ is unbiased

Assume another linear estimator $\underset{1\times N}{c^T}\, \underset{N\times 1}{\underset{\sim}{y}}$

Unbiased means here: $E(c^T y) = a^T\beta$ for all β

$\Updownarrow$

$c^T \underset{\sim}{X}\beta = a^T\beta$ for all β

$\Downarrow$ since $E(y) = \underset{\sim}{X}\beta$

Requirement for unbiasedness $\rightarrow$ $c^T \underset{\sim}{X} = a^T$ (*)

since the above is for all β

Now $Var(a^T\hat{\beta}) = a^T(\underset{\sim}{X}^T\underset{\sim}{X})^{-1} a\, \sigma^2$

$= c^T \underset{\sim}{X}(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{X}^T c\, \sigma^2$
use (*)

$= c^T H c\, \sigma^2$

while $Var(c^T y) = c^T\sigma^2 I c = c^T c\, \sigma^2$

Difference in variances is

$Var(c^T y) - Var(a^T\hat{\beta}) = c^T(I - H)c\,\sigma^2$

since H is idempotent, i.e. $H = H^2$

$$= c^T (I-H)^2 c \sigma^2 = \left(c^T (I-H)\right)\left((I-H) c\right) \sigma^2$$

$$= d^T d \quad \text{where } d^T = c^T (I-H)$$

This is ≥ 0, and $= 0$ only if $d = 0$, i.e.

$$c^T (I-H) = 0$$

or $\quad c^T = c^T H$

But in this case

$$c^T y = c^T H y = c^T X \hat{\beta} = a^T \hat{\beta}$$

so the two estimators are equal

Hence $\mathrm{Var}(c^T y) \geq \mathrm{Var}(a^T \hat{\beta})$ with equality if and only if the two estimators are equal.