Problem 1:

a) \( d(t; x_1, x_2) = \lambda_0(t) e^{\beta_1 x_1 + \beta_2 x_2} \)

Assumptions:
\( \lambda_0(t) \) arbitrary function, \( \geq 0 \)
Potential lifetimes independent
Independent censoring

Mainly highway: \((x_1, x_2) = (0, 1)\)

so \( d(t; 0, 1) = \lambda_0(t) e^{\beta_2} \)

and hence \( S(t; 0, 1) = e^{-\lambda_0(t) e^{\beta_2}} \)

\[ L(\beta_1, \beta_2) = \prod_{i=1}^{3} \frac{e^{\beta_1 x_{1i} + \beta_2 x_{2i}}}{\sum_{j \in R(i)} e^{\beta_1 x_{1j} + \beta_2 x_{2j}}} \]

(3 lifetimes in data)

\[ = \frac{1}{2 + 2e^{\beta_1} + 2e^{\beta_2}} \cdot \frac{e^{\beta_2}}{1 + e^{\beta_1} + 2e^{\beta_2}} \cdot \frac{e^{\beta_1}}{e^{\beta_1} + e^{\beta_2}} \]
c) \( l(\hat{\beta}_1, \hat{\beta}_2) = \ln L(\beta_1, \beta_2) = -3.67 \)

\( l(0, 0) = \ln L(0, 0) = \ln \frac{1}{6.42} \)

\( = \ln \frac{1}{48} = -\ln 48 = -3.87 \)

Want to test \( H_0: \beta_1 = \beta_2 = 0 \) vs. \( H_1: \text{not so} \)

Test statistic: \( W = 2 \left( l(\hat{\beta}_1, \hat{\beta}_2) - l(0, 0) \right) \)

\( = 2(-3.67 - (-3.87)) \)

\( = 0.40 \)

Under \( H_0 \) is \( W \sim \chi^2_2 \). Thus no reason to reject \( (\text{reject for } W \geq 5.991) \)

Problem 2

a) \( W(t) \) estimated by \( N(t) = \# \text{events before } t \) \n(special case of Nelson-Aalen estimator).
\[ W(t) = N(t) \]

Indicates increasing trend (convex plot)

\[ \hat{W}(595) = 5 \]

\[ \text{SD} \hat{W}(595) = \sqrt{5} \] (standard estimator of variance for NTPP)

Standard interval for \( \hat{W}(595) \):

\[ 5 \pm 1.96 \sqrt{5} \]

\[ (0.617, 9.38) \]

Standard positive interval (based on \( \ln \hat{W}(595) \) approx. normal):

\[ 5 \cdot e ^ { \pm 1.96 \frac{\sqrt{5}}{5} } \]

\[ (2.08, 12.01) \]
b) Laplace test with time truncation:

\[ L = \frac{\sum S_i - n c \cdot 19.94 - 5.650}{\sqrt{\frac{n c^2}{12}}} = \frac{5.650^2}{12} = 0.8795 \]

P-value for testing \( H_0: \) no trend vs. \( H_1: \) trend (two-sided test) is \( z \)

\[ 2 \cdot P(Z \geq 0.8795) = 0.38 \]

(\( N(0, 1) \))

Conclusion: Do not reject \( H_0. \)

Positive sign means tendency of increasing trend (compare with a).

c) \( W(t) = \int w(u) \, du = \int e^{-\lambda u} \, du \)

\[ = e^{-\lambda} \left[ \frac{e^{\beta u}}{\beta} \right]_0^t = \frac{e^{\beta t} - 1}{\beta} \]
\[ L(\alpha, \beta) = \frac{1}{n} \sum_{i=1}^{n} \omega(z_i) e^{-W(z_i)} \]

\[ \ell(\alpha, \beta) = \sum_{i=1}^{n} \ln \omega(z_i) - W(z_i) \]

\[ = \sum_{i=1}^{n} \left( \alpha + \beta z_i \right) - \frac{e^\alpha (e^{\beta z_i} - 1)}{\beta} \]

\[ = 5\alpha + \beta \cdot 1994 - \frac{e^\alpha (e^{650\beta} - 1)}{\beta} \]

\[ d) \quad \hat{\ell}(\beta) = \ell(\hat{\alpha}(\beta), \beta) \]

where \( \hat{\alpha}(\beta) \) maximizes \( \ell(\alpha, \beta) \) for fixed \( \beta \).

\[ \frac{\partial \ell(\alpha, \beta)}{\partial \alpha} = 5 - \frac{e^\alpha (e^{650\beta} - 1)}{\beta} = 0 \]

\[ \Rightarrow \quad e^{\hat{\alpha}(\beta)} = \frac{5\beta}{e^{650\beta} - 1} \]

\[ \ell(\beta) = \ln \frac{5\beta}{e^{650\beta} - 1} \]
\[ \ell(\beta) = 5 \ln \frac{5/n}{\cos \beta - 1} + 1994\beta - 5 \]

1. Read off maximum: \[ \beta = 0.0022 \]

2. 95% confidence:

\[ \{ \beta : \ell(\beta) = \ell(\hat{\beta}) - 1.92 \} \]

\[ -30.85 \]

since \[ \ell(\beta) = -28.93 \]

(read off!)

**Approximate reading:**

95% confidence: \((-0.003, 0.008)\)

3. \(H_0: \beta = 0 \) vs \(H_1: \beta \neq 0\).

Test statistic \(W = 2(\ell(\beta) - \ell(0))\)

\[ = 2(-28.93 - (-29.33)) \]

\[ = 0.80 \]

\(W \times \chi^2_1 \) under \(H_0\).

Thus do not reject since this is \(\leq 3.84\)

Or: Do not reject since \(0 \in 95\%\) CI from \(\ell(\beta)\).
e) Use that events in disjoint intervals are independent and Poisson.

\[ L(\alpha, \beta) = \frac{(W(200) - W(100))'}{1!} \frac{(W(400) - W(300))'}{1!} \]

\[ \frac{(W(500) - W(400))'}{1!} \frac{(W(600) - W(500))}{2!} \]

\[ e^{-W(650)} \cdot e \]

\[ \frac{1}{2} \left( \frac{e^\alpha}{\beta} \right)^5 \cdot (e^{200\beta} - e^{100\beta}) \cdot (e^{400\beta} - e^{300\beta}) \]

\[ \cdot (e^{500\beta} - e^{400\beta}) \cdot (e^{600\beta} - e^{500\beta})^2 \]

\[ e^{-e^{650\beta}} \cdot e^{-\frac{\alpha}{\beta} \cdot (e^{650\beta} - 1)} \]
Problem 3

a) Let $\ln T = \mu + \sigma \, \ln \gamma$

or $T = e^{\mu + \sigma \, \ln \gamma}$

This defines a lifetime distribution

where

$$F_T(t) = \Phi_0 \left( \frac{\ln t - \mu}{\sigma} \right)$$

$$f_T(t) = \phi_0 \left( \frac{\ln t - \mu}{\sigma} \right) \cdot \frac{1}{\sigma t}$$

b) First note that

$$1 - \Phi_0(t) = \frac{1}{1 + e^u}$$

so that

$$R_T(t) = \frac{1}{1 + e^{\frac{\ln t - \mu}{\sigma}}} = \frac{1}{1 + e^{\frac{\alpha}{\theta}}}$$

$$= \frac{1}{1 + e^{-\alpha}} \quad \text{if} \quad \alpha = \frac{1}{\theta}$$

$$\delta = e^{-\frac{\alpha}{\theta}}.$$
Probability plot:

\[ R_T(t) = 1 + \gamma t \leq \alpha \]

\[ \gamma t \leq \alpha = R_T(t) = 1 - 1 \]

\[ \ln(\gamma + \alpha) \ln(t) = \ln(R_T(t) = 1 - 1) \]

Thus as \((\ln(t), \ln(R_T(t) = 1 - 1))\) as \(t\) varies, is on a straight line; we can plot for the non-censored \(t_i\):

\[ (\ln(t_i), \ln(R_T(t_i) = 1 - 1)) \]

where \(R_T(.\) is an estimate of \(R_T(.\) based only on the data (Kaplan-Meier or a modification of it).

c) Write

\[ \ln(T) = \beta_0 + \beta_1 x + \cdots + \beta_k x_k + \sigma W \]

\[ = \beta^T x + \sigma W \]

Thus we let \( \mu = \beta^T x \) in the above B). Hence \( Y = e^{-\frac{\mu}{\sigma}} = e^{-\alpha^T x} \) and we set

\[ R_T(t; x) = \frac{1}{1 + e^{-\alpha^T x t}} \]