TMA4275 V05
Exam 04.06.2005

Problem 1 :

a) $\lambda(t; x_1, x_2) = \lambda_0(t) e^{\beta_1 x_1 + \beta_2 x_2}$

Assumptions:

$\lambda_0(t)$ arbitrary function, ≥ 0
Potential lifetimes independent
Independent censoring

Mainly higway : $(x_1, x_2) = (0, 1)$

so $\lambda(t; 0, 1) = \lambda_0(t) e^{\beta_2}$

and hence

$$\underline{\underline{S(t; 0,1) = e^{-\Lambda_0(t) e^{\beta_2}}}}$$

b) $$L(\beta_1, \beta_2) = \prod_{i=1}^{3} \frac{e^{\beta_1 x_{l_i 1} + \beta_2 x_{l_i 2}}}{\sum_{j \in R(t_i)} e^{\beta_1 x_{j1} + \beta_2 x_{j2}}}$$

(3 <u>lifetimes</u> in data)

$$\underline{\underline{= \frac{1}{2 + 2e^{\beta_1} + 2e^{\beta_2}} \cdot \frac{e^{\beta_2}}{1 + e^{\beta_1} + 2e^{\beta_2}} \cdot \frac{e^{\beta_1}}{e^{\beta_1} + e^{\beta_2}}}}$$

c) $l(\hat{\beta}_1, \hat{\beta}_2) = \ln L(\hat{\beta}_1, \hat{\beta}_2) = -3.67$

$$l(0,0) = \ln L(0,0) = \ln \frac{1}{6 \cdot 4 \cdot 2}$$

$$= \ln \frac{1}{48} = -\ln 48 = -3.87$$

Want to test $H_0: \beta_1 = \beta_2 = 0$ vs. H_1: not so

Test statistic: $W = 2\left(l(\hat{\beta}_1, \hat{\beta}_2) - l(0,0)\right)$

$$= 2(-3.67 - (-3.87))$$

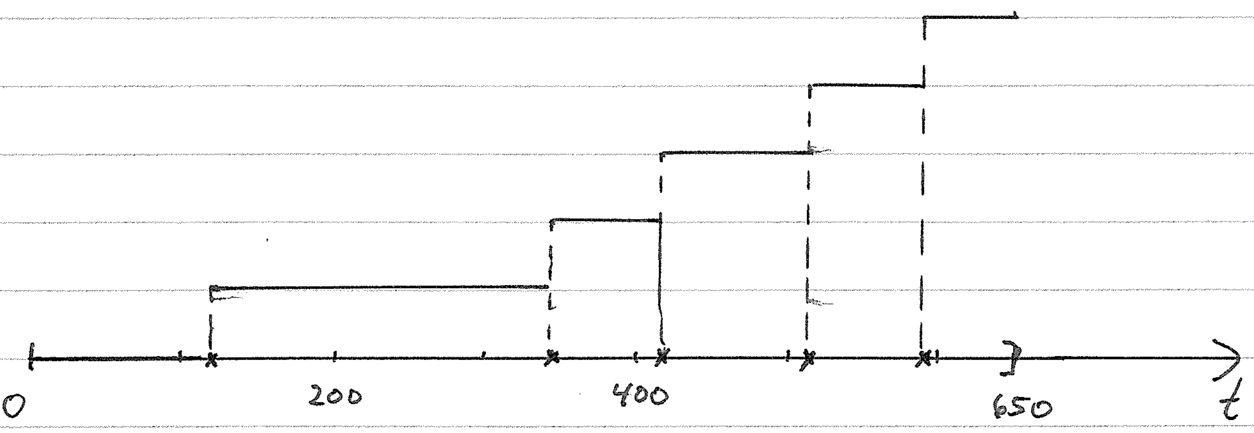
$$= 0.40$$

Under H_0 is $W \approx \chi^2_2$. Thus no reason to reject (reject for $W \geq 5.991$)

Problem 2

a) $W(t)$ estimated by $N(t) = \#$ events before t (special case of Nelson-Aalen estimator).

$\hat{W}(t) = N(t)$



Indicates increasing trend (convex plot)

$\hat{W}(595) = 5$

$\widehat{SD\,\hat{W}(595)} = \sqrt{5}$ (standard estimator of variance for NHPP)

Standard interval for $W(595)$: $5 \pm 1.96\sqrt{5}$

$$\underline{\underline{(0.617,\ 9.38)}}$$

Standard positive interval (based on $\ln \hat{W}(595)$ approx. normal):

$$5 \cdot e^{\pm 1.96 \frac{\sqrt{5}}{5}}$$

$$\underline{\underline{(2.08,\ 12.01)}}$$

b) Laplace test with time truncation:

$$L = \frac{\sum_{i=1}^{n} S_i - \frac{n\tau}{2}}{\sqrt{\frac{n\tau^2}{12}}} = \frac{1994 - \frac{5 \cdot 650}{2}}{\sqrt{\frac{5 \cdot 650^2}{12}}} = \underline{0.8795}$$

P-value for testing H_0: no trend
vs. H_1: trend
(two-sided test) is $\approx$

$$2 \cdot P(\underset{N(0,1)}{Z} \geq 0.8795) = \underline{0.38}$$

Conclusion: Do <u>not</u> reject H_0.

Positive sign means tendency of increasing trend (compare with (a)).

c) $$W(t) = \int_0^t w(u)\,du = \int_0^t e^{\alpha+\beta u}\,du$$

$$= e^{\alpha} \left| \frac{e^{\beta u}}{\beta} \right|_0^t = \frac{e^{\alpha}(e^{\beta t}-1)}{\beta}$$

$$L(\alpha,\beta) = \prod_{i=1}^{5} w(s_i) e^{-W(z)}$$

$$\ell(\alpha,\beta) = \sum_{i=1}^{5} \ln w(s_i) - W(z)$$

$$= \sum_{i=1}^{5} (\alpha + \beta s_i) - \frac{e^{\alpha}(e^{\beta z} - 1)}{\beta}$$

$$= 5\alpha + \beta \cdot 1994 - \frac{e^{\alpha}(e^{650\beta} - 1)}{\beta}$$

d) $\tilde{\ell}(\beta) = \ell(\hat{\alpha}(\beta), \beta)$

where $\hat{\alpha}(\beta)$ maximizes $\ell(\alpha,\beta)$ for fixed β.

$$\frac{\partial \ell(\alpha,\beta)}{\partial \alpha} = 5 - \frac{e^{\alpha}(e^{650\beta} - 1)}{\beta} = 0$$

$$\Rightarrow e^{\hat{\alpha}(\beta)} = \frac{5\beta}{e^{650\beta} - 1}$$

$$\hat{\alpha}(\beta) = \ln \frac{5\beta}{e^{650\beta} - 1}$$

$$\Rightarrow \tilde{\ell}(\beta) = 5 \ln \frac{5\beta}{e^{650\beta} - 1} + 1994\beta - 5$$

1. Read off maximum: $\hat{\beta} = 0.0022$

2. 95% conf. int.

$$\{\beta : \tilde{\ell}(\beta) \geq \underbrace{\tilde{\ell}(\hat{\beta}) - 1.92}_{-30.85}\}$$

since $\tilde{\ell}(\hat{\beta}) = -28.93$
(read off!)

Approximate reading:
95% conf. int: $(-0.003, 0.008)$

3. $H_0: \beta = 0$ vs $H_1: \beta \neq 0$.

Test statistic $W = 2(\tilde{\ell}(\hat{\beta}) - \tilde{\ell}(0))$

$= 2(-28.93 - (-29.33))$

$= 0.80$

$W \approx \chi^2_1$ under H_0.
Thus do not reject since this is ≤ 3.84

Or: Do not reject since $0 \in$ 95% CI from "2".

e) Use that #events in disjoint intervals are independent and Poisson.

$$L(\alpha,\beta) = \frac{(W(200)-W(100))^1}{1!} \cdot \frac{(W(400)-W(300))^1}{1!}$$

$$\cdot \frac{(W(500)-W(400))^1}{1!} \cdot \frac{(W(600)-W(500))^2}{2!}$$

$$\cdot e^{-W(650)}$$

$$= \frac{1}{2}\left(\frac{e^{\alpha}}{\beta}\right)^5 \cdot \left(e^{200\beta} - e^{100\beta}\right) \cdot \left(e^{400\beta} - e^{300\beta}\right)$$

$$\cdot \left(e^{500\beta} - e^{400\beta}\right) \cdot \left(e^{600\beta} - e^{500\beta}\right)^2$$

$$\cdot e^{-\frac{e^{\alpha}(e^{650\beta}-1)}{\beta}}.$$

Problem 3

a) Let $\ln T = \mu + \sigma W$

or $T = e^{\mu + \sigma W}$

This defines a lifetime distribution where

$$F_T(t) = \Phi_0\left(\frac{\ln t - \mu}{\sigma}\right)$$

$$f_T(t) = \varphi_0\left(\frac{\ln t - \mu}{\sigma}\right) \cdot \frac{1}{\sigma t}$$

b) First note that

$$1 - \Phi_0(t) = \frac{1}{1 + e^{w}}$$

so that

$$R_T(t) = \frac{1}{1 + e^{\frac{\ln t - \mu}{\sigma}}} = \frac{1}{1 + t^{\frac{1}{\sigma}} e^{-\frac{\mu}{\sigma}}}$$

$$= \frac{1}{1 + \gamma t^{\alpha}} \quad \text{if } \alpha = \frac{1}{\sigma}$$

$$\gamma = e^{-\frac{\mu}{\sigma}}.$$

<u>Probability plot:</u>

$$R_T(t)^{-1} = 1+\gamma t^{\alpha}$$

$$\gamma t^{\alpha} = R_T(t)^{-1} - 1$$

$$\ln \gamma + \alpha \ln t = \ln\left(R_T(t)^{-1} - 1\right)$$

Thus as $\left(\ln t, \ln\left(R_T(t)^{-1} - 1\right)\right)$ as t varies is on a straight line; we can plot for the noncensored $t_{(i)}$:

$$\left(\ln t_{(i)}, \ln\left(\hat{R}_T(t_{(i)})^{-1} - 1\right)\right)$$

where $\hat{R}_T(\cdot)$ is an estimate of $R_T(\cdot)$ based only on the data (Kaplan-Meier or a modification of it).

c) Write

$$\ln T = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + \sigma W$$

$$\equiv \beta' x + \sigma W$$

Thus we let $\mu = \beta' x$ in the above b). Hence $\gamma = e^{-\frac{\mu}{\sigma}} = e^{-\alpha\beta' x}$ and we get

$$R_T(t; x) = \frac{1}{1+e^{-\alpha\beta' x}\cdot t^{\alpha}}$$