EXAM TMA4275 MAY 2006

Exercise 1

a) Solve $R(t;\alpha,\theta) = 1-p$

$$1+\left(\frac{t}{\theta}\right)^\alpha = \frac{1}{1-p}$$

$$\left(\frac{t}{\theta}\right)^\alpha = \frac{p}{1-p}$$

$$\alpha \ln t - \alpha \ln\theta = \ln\frac{p}{1-p}$$

$$\underline{\ln t = \ln\theta + \frac{1}{\alpha}\ln\frac{p}{1-p}}$$

so $\underline{\underline{\ln t_p(\alpha,\theta) = \ln\theta + \frac{1}{\alpha}\ln\frac{p}{1-p}}}$

Median: $p = 1/2$

$$\ln t_{1/2}(\alpha,\theta) = \ln\theta$$

$\Rightarrow \underline{\underline{t_{1/2}(\alpha,\theta) = \theta}}$ (median)

Q1: $\ln t_{1/4}(\alpha,\theta) = \ln\theta + \frac{1}{\alpha}\ln\frac{1}{3}$

$$\Rightarrow \underline{\underline{t_{1/4}(\alpha,\theta) = e^{\ln\theta + \frac{1}{\alpha}\ln\frac{1}{3}} = \theta\left(\frac{1}{3}\right)^{1/\alpha}}}$$

Q3: $\ln t_{3/4}(\alpha,\theta) = \ln\theta + \frac{1}{\alpha}\ln 3$

$$\Rightarrow \quad \underline{\underline{t_{3/4}(\alpha,\theta) = \theta\cdot 3^{\frac{1}{\alpha}}}}$$

b) Compute first

$$f(t;\alpha,\theta) = -R'(t;\alpha,\theta)$$

$$= \frac{\frac{\alpha}{\theta}\cdot\left(\frac{t}{\theta}\right)^{\alpha-1}}{\left\{1+\left(\frac{t}{\theta}\right)^{\alpha}\right\}^2}$$

Now $z(t;\alpha,\theta) = \dfrac{f(t;\alpha,\theta)}{R(t;\alpha,\theta)}$

$$= \frac{\frac{\alpha}{\theta}\cdot\left(\frac{t}{\theta}\right)^{\alpha-1}}{1+\left(\frac{t}{\theta}\right)^{\alpha}} = \underline{\underline{\frac{\alpha t^{\alpha-1}}{\theta^{\alpha}+t^{\alpha}}}}$$

$$z'(t;\alpha,\theta) = \frac{\alpha(\alpha-1)t^{\alpha-2}(\theta^{\alpha}+t^{\alpha}) - \alpha t^{\alpha-1}\cdot\alpha t^{\alpha-1}}{(\theta^{\alpha}+t^{\alpha})^2}$$

$$= \frac{\alpha t^{\alpha-2}\left[(\alpha-1)\theta^{\alpha}+(\alpha-1)t^{\alpha}-\alpha t^{\alpha}\right]}{(\theta^{\alpha}+t^{\alpha})^2}$$

$$= \frac{\alpha t^{\alpha-2}\left[(\alpha-1)\theta^{\alpha}-t^{\alpha}\right]}{(\theta^{\alpha}+t^{\alpha})^2}$$

This is decreasing iff $(\alpha-1)\theta^\alpha - t^\alpha < 0$

$$\Updownarrow$$

$$t^\alpha > (\alpha-1)\theta^\alpha$$

If $\alpha \le 1$: This holds for all $t > 0$.
If $\alpha > 1$: " " iff $t > \theta\cdot(\alpha-1)^{1/\alpha}$

Thus: For $\alpha > 1$: $z(t;\alpha,\theta)$ $\nearrow$ for $t < \theta\cdot(\alpha-1)^{1/\alpha}$
$\searrow$ for $t > \theta\cdot(\alpha-1)^{1/\alpha}$

Reasonable for repair times (see book under lognormal distr.).

c) $1 - F_Y(y) = P(Y > y) = P(\ln T > y)$

$$= P(T > e^y) = \frac{1}{1+\left(\frac{e^y}{\theta}\right)^\alpha} = \frac{1}{1+e^{y\alpha - \alpha\ln\theta}}$$

$$= \frac{1}{1+e^{\frac{y-\ln\theta}{1/\alpha}}} = 1 - \Phi_0\left(\frac{y-\ln\theta}{1/\alpha}\right)$$

if $1-\Phi_0(w) = \dfrac{1}{1+e^w}$ so $\Phi_0(w) = \dfrac{e^w}{1+e^w}$

Thus $\mu = \ln\theta$, $\sigma = 1/\alpha$.
Φ_0 is the cdf of standard logistic distr.

so T has a log-logistic distribution..

Exercise 2

a) Failures at 212, 445, 792

$$\hat{R}(212) = \frac{\text{numb. at risk} - 1}{\text{numb at risk}} = \frac{10}{11} = \underline{0.9091}$$

$$\hat{R}(445) = \frac{10}{11} \cdot \frac{n_2 - 1}{n_2} = \frac{10}{11} \cdot \frac{9}{10} = \frac{9}{11} = \underline{0.8182}$$

$$\hat{R}(792) = \frac{10}{11} \cdot \frac{9}{10} \cdot \frac{8}{9} = \frac{8}{11} = \underline{0.7273}$$

Conditions: ① Lifetimes are independent from $R(t)$

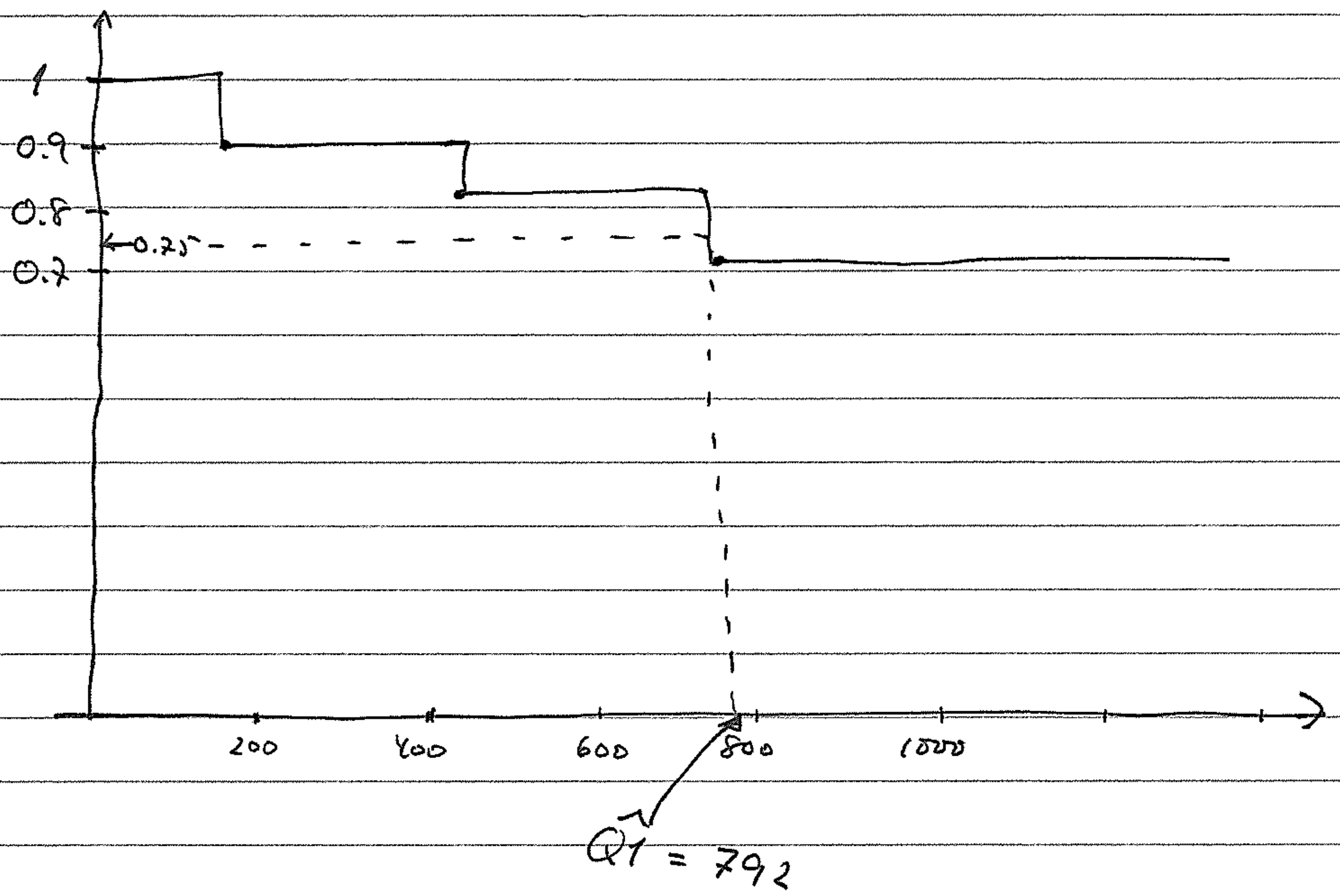
② Independent censoring.

$$Q1 = t_{0.25} = R^{-1}(0.75)$$

Estimate by $\underline{\underline{\hat{R}^{-1}(0.75) = 792}}$

Cannot estimate median and $Q3$ because KM-estimates do not go under 0.7273.

Diagram:



b) Since $\alpha, \theta > 0$ will we use the standard interval for positive parameters :

$$\alpha: \quad \hat{\alpha} e^{\pm 1.96 \cdot \frac{\widehat{SD}(\hat{\alpha})}{\hat{\alpha}}}$$

$$1.2868 \cdot e^{\pm 1.96 \frac{\sqrt{0.44512}}{1.2868}}$$

$$(0.4658,\ 3.5550)$$

θ: $\quad \hat{\theta} \cdot e^{\pm 1.96 \frac{\widehat{SD(\hat{\theta})}}{\hat{\theta}}}$

$$2286.5 \cdot e^{\pm 1.96 \cdot \frac{\sqrt{2.6638 \cdot 10^3}}{2286.5}}$$

$$\underline{\underline{(564.38,\ 9263.5)}}$$

$$\hat{Q1} = \hat{\theta}\left(\frac{1}{3}\right)^{1/\hat{\alpha}} = 2286.5 \cdot \left(\frac{1}{3}\right)^{1/1.2868} = \underline{\underline{973.62}}$$

From (a) $\quad$ (compare to 792, nonparametric)

Median and Q3 can be estimated since we have a parametric model over $(0, \infty)$.

$$\hat{Q3} = \hat{\theta} \cdot 3^{1/\hat{\alpha}} = 2286.5 \cdot 3^{1/1.2868} = \underline{\underline{5369.7}}$$

$$\text{Median} = \theta \Rightarrow \widehat{\text{Median}} = \hat{\theta} = \underline{\underline{2286.5}}$$

(c) $\ln\left(\frac{1 - R(t; \alpha, \theta)}{R(t; \alpha, \theta)}\right)$

$$= \ln\left(\frac{\frac{\left(\frac{t}{\theta}\right)^\alpha}{1+\left(\frac{t}{\theta}\right)^\alpha}}{\frac{1}{1+\left(\frac{t}{\theta}\right)^\alpha}}\right) = \ln\left(\frac{t}{\theta}\right)^\alpha = \alpha \ln t - \alpha \ln \theta$$

Therefore: $\left(\ln t,\ \ln\left(\frac{1 - R(t; \alpha, \theta)}{R(t; \alpha, \theta)}\right)\right)$

are on the line "$y = \alpha x - \alpha \ln \theta$"

Probability plot:

For failure times $t_{(1)} < t_{(2)} < \cdots$

plot

$$\left(\ln t_{(i)},\ \ln \frac{1-\hat{\hat{R}}(t)}{\hat{\hat{R}}(t)}\right)$$

for an estimate $\hat{\hat{R}}(t)$ of $\hat{R}(t)$.

Should be compared to line (estimated)

$$\text{"}y = 1.2868x - 1.2868 \cdot \ln 2286.5$$

i.e. $\text{"}y = 1.2868x - 9.9531$

$t_{(1)} = 212$: $\hat{R}(212) = \frac{10}{11}$, $\hat{\hat{R}}(212) = \frac{1 + \frac{10}{11}}{2} = \frac{21}{22}$

Modified KM

$\ln \frac{1-\hat{R}(t_{(1)})}{\hat{R}(t_{(1)})} = -2.3$, $\ln \frac{1-\hat{\hat{R}}(t_{(1)})}{\hat{\hat{R}}(t_{(1)})} = -3.0$

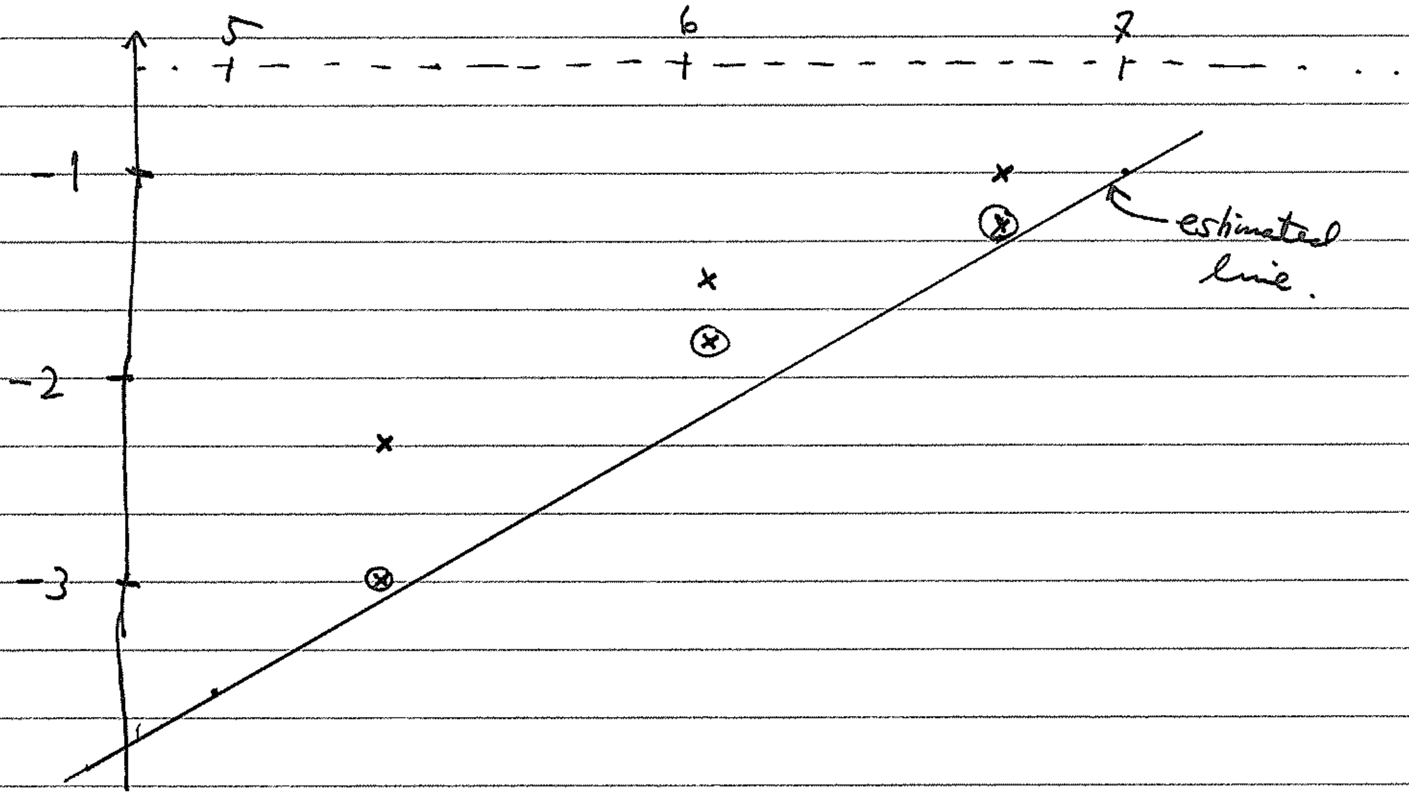
$t_{(2)} = 445$: $\hat{R}(445) = \frac{9}{11}$, $\hat{\hat{R}}(445) = \frac{\frac{10}{11} + \frac{9}{11}}{2} = \frac{19}{22}$

$\ln \frac{1-\hat{R}(445)}{\hat{R}(445)} = -1.5$, $\ln \frac{1-\hat{\hat{R}}(445)}{\hat{\hat{R}}(445)} = -1.8$

$t_{(3)} = 792$: $\hat{R}(792) = \frac{8}{11}$, $\hat{\hat{R}}(792) = \frac{\frac{9}{11} + \frac{8}{11}}{2} = \frac{17}{22}$

$\ln \frac{1-\hat{R}(792)}{\hat{R}(792)} = -1.0$, $\ln \frac{1-\hat{\hat{R}}}{\hat{\hat{R}}} = -1.2$

Plot:



$\ln 212 = 5.4$ ⊗ Modified KM

$\ln 445 = 6.1$ × KM

$\ln 792 = 6.7$

Modified KM gives a very good fit to the line.

Exercise 3

Nelson-Aalen:

$$\hat{W}(t) = \sum_{\substack{t_i \leq t \\ \uparrow}} \frac{d_i}{Y(t_i)}.$$

Projected failure times

where $t_1 < t_2 < \cdots$ are points of failure in any system,

d_i = # failing at t_i

$Y(t_i)$ = # systems at t_i

Here: $Y(t_i) \equiv 7$ so

$$\hat{W}(t) = \frac{1}{7} \cdot \{\text{\# failures at or before } t\}$$

(failures = goals in this exercise).

Note: $d = 3$ at time 63, otherwise $d = 1$.

$W(45) = E(N(45))$ = expected numb of failures (goals) before 45 min (1st half of football match).

$$\hat{W}(45) = \frac{\text{\# fail. before or at 45}}{7} = \frac{4}{7} = \underline{\underline{0.57}}$$

$$\widehat{\operatorname{Var}} \hat{W}(t) \underset{\text{for NHPP's}}{=} \sum_{t_i \le t} \frac{d_i}{Y(t_i)^2} \underset{\text{here}}{=} \frac{1}{7^2} \sum_{t_i \le t} d_i$$

So

$$\widehat{\operatorname{Var}} \hat{W}(45) = \frac{1}{7^2} \cdot 4 = \frac{4}{49}$$

$$\widehat{SD}\, \hat{W}(45) = \sqrt{\frac{4}{49}} = \frac{2}{7} = \underline{\underline{0.29}}$$

b) MINITAB – plot: Seems to be an increasing derivative of $W(t)$ from 80 minutes. (Thus $w(t)$ seems to be ↗ for large t).

From data: $H_0: w(t) = \lambda$ (constant) vs. $H_1: w(t) \nearrow$

$$Z_{pooled} = 2 \cdot \sum_{\text{all } t_i} \ln \frac{90}{t_i}$$

$$= 2 \left[\ln \frac{90}{9} + \ln \frac{90}{63} + \ln \frac{90}{78} + \cdots + \ln \frac{90}{13} \right]$$

$$= 16.01$$

Under H_0 is $Z_{pooled} \sim \chi^2_{2 \cdot 12} = \chi^2_{24}$

Reject H_0 if $Z_{pooled} < \chi_{0.95, 24} = 13.85$

so we do not reject!

TTT-based test is obtained by projecting all times to one axis, then multiplying every number by 7 (constant # of systems for all times). Thus $Z_{TTT} = 2 \cdot$ sum of $\ln \frac{7 \cdot old}{7 \cdot old}$ so gets the same!

Reason: τ (censoring time) is same for all processes.

c) General (log) likelihood expression:

$$\ell(\theta) = \sum_{j=1}^{m} \left[\sum_{i=1}^{N_j} \ln w(s_{ij} ; \theta) - W(\tau_j, \theta) \right]$$

Here: $w(t) = \lambda$

$W(t) = \lambda t$

$\tau_j = \tau = 90$

$$\ell(\lambda) = \sum_{j=1}^{m} \left[\sum_{i=1}^{N_j} \ln \lambda - \lambda \tau \right]$$

$$= 12 \ln \lambda - 630 \lambda$$

$$\ell'(\lambda) = \frac{12}{\lambda} - 630$$

$$\ell''(\lambda) = -\frac{12}{\lambda^2}$$

Thus: $\ell'(\hat{\lambda}) = 0 \Rightarrow \hat{\lambda} = \frac{12}{630} = \underline{0.019}$

$$\widehat{SD(\hat{\lambda})} = \sqrt{\frac{1}{-\ell''(\hat{\lambda})}} = \sqrt{\frac{\hat{\lambda}^2}{12}} = \frac{\hat{\lambda}}{\sqrt{12}}$$

$$= \underline{\underline{0.0055}}$$