ST 2202 5. Desember 2008

Oppgave 1:

a) La $f(x,y) = 2\ln x - \frac{1}{2}\ln y$

En Taylor-utvikling om $(x_0, y_0)$ gir da

$$f(x,y) \approx f(x_0,y_0) + \frac{2}{x_0}(x-x_0) - \frac{1}{2y_0}(y-y_0)$$

$\Rightarrow$ Med $x_0 = E(X) = e^{\mu+\frac{1}{2}\sigma^2}$, $y_0 = E(X^2) = e^{2\mu+2\sigma^2}$

$$\hat{\mu} \approx 2(\mu+\tfrac{1}{2}\sigma^2) - \tfrac{1}{2}(2\mu+2\sigma^2)$$

$$+ \frac{2}{e^{\mu+\frac{1}{2}\sigma^2}}(\overline{X}_n - e^{\mu+\frac{1}{2}\sigma^2})$$

$$- \frac{1}{2e^{2\mu+2\sigma^2}}(Y_n - e^{2\mu+2\sigma^2})$$

$$= \mu + 2e^{-\mu-\frac{1}{2}\sigma^2}(\overline{X}_n - e^{\mu+\frac{1}{2}\sigma^2})$$

$$- \frac{1}{2}e^{-2\mu-2\sigma^2}(Y_n - e^{2\mu+2\sigma^2})$$

$$\Rightarrow Var(\hat{\mu}) \approx 4e^{-2\mu-\sigma^2} \cdot \frac{e^{2\mu+\sigma^2}(e^{\sigma^2}-1)}{n}$$

$$+ \frac{1}{4}e^{-4\mu-4\sigma^2} \cdot \frac{e^{4\mu+4\sigma^2}(e^{4\sigma^2}-1)}{n}$$

$$-2e^{-3\mu-\frac{5}{2}\sigma^2} \cdot \frac{e^{3\mu+\frac{5}{2}\sigma^2}(e^{2\sigma^2}-1)}{n}$$

$$= \frac{4(e^{\sigma^2}-1) + \frac{1}{4}(e^{4\sigma^2}-1) - 2(e^{2\sigma^2}-1)}{n}$$

$$= \frac{4e^{\sigma^2} + \frac{1}{4}e^{4\sigma^2} - 2e^{2\sigma^2} - \frac{9}{4}}{n}$$

$$= \frac{16e^{\sigma^2} + e^{4\sigma^2} - 8e^{2\sigma^2} - 9}{4n}$$

$$\hat{\mu} = 2\ln\frac{104.0}{10} - \frac{1}{2}\ln\frac{1230.44}{10}$$

$$= \underline{2.2773}$$

$$\hat{\sigma}^2 = \ln\frac{1230.44}{10} - 2\ln\frac{104.0}{10}$$

$$= \underline{0.1289}, \quad e^{\hat{\sigma}^2} = 1.1376$$

$$Var(\hat{\mu}) \simeq \frac{16 \cdot 1.1376 + 1.1376^4 - 8 \cdot 1.1376^2 - 9}{40}$$

$$= 0.0131 \quad (= 0.1144^2)$$

b) Trenger

$$\hat{\mu}^{(1)} = 2\ln\frac{1040-8.2}{9} - \frac{1}{2}\ln\frac{123044-8.2^2}{9}$$

$$= 2.299$$

$$\Rightarrow \quad T_n^{(\cdot)} = \frac{20.477+2.299}{10} = \underline{2.2776}$$

$$\left(\widehat{\mathrm{Var}(\hat{\mu})}\right)_{\text{Jack}} = \frac{9}{10}\sum_{i=1}^{10}\left(T_n^{(i)} - T_n^{(\cdot)}\right)^2$$

$$= \frac{9}{10}\cdot\left[\sum_{i=1}^{10}\left(T_n^{(i)\,2}\right) - 10\cdot T_n^{(\cdot)\,2}\right]$$

$$= \frac{9}{10}\cdot\left[46.604987 + 2.299^2 - 10\cdot 2.2776^2\right]$$

$$= \underline{\underline{0.0142}} = \underline{\underline{0.1191^2}}$$

c) Dette er standard fra pensum

d) Tegntest: $H_0: m = 9.0$ mot $H_1: m > 9.0$.

Ta bort observ. 9.0.
$X_+$ = antall som er $> 9.0$ er 6
$n$ = tot. ant. $= 9$.

P-verdi: $P(X_+ \geq 6)$ da $X_+ \sim \text{bin}(9, \frac{1}{2})$

$$= 1 - P(X_+ \leq 5)$$

$$= 1 - 0.7461 = \underline{\underline{0.2539}}$$

Så – forkaste ikke på nivå 5%.

## Oppgave 2

$\hat{A}$ = average of ~~A~~ Y when A is +1
 − average of Y when A is −1

$$= \frac{70.8+73.2+79.3+91.2}{4} - \frac{69.3+71.3+77.5+88.9}{4}$$

$= \underline{1.8750}$ (MINITAB)

Alias-structure shows that this estimate is confounded with BD, CE, ABCDE

$\widehat{BC}$ = compute by contrast ±1

$$= \frac{Y_1+Y_2-Y_3-Y_4-Y_5-Y_6+Y_7+Y_8}{4} = \underline{4.7250}$$

Confounded with DE, ABE, ACD.

Defining relation: From D = AB, E = AC we get

I = ABD, I = ACE,
I = ABD·ACE = BCDE

so defining relation is
$\underline{I = ABD = ACE = BCDE}$

⟹ Resolution is 3 (lowest number of letters)

Practical interpretation: Resolution $k \Rightarrow$ main effects are confounded with effects of order $k-1$; second order interactions are confounded with order $k-2$ etc.

b) It follows from alias-structure, by setting all interactions with D or E equal to 0; that the alias structure now is

I
A
B
C
D + AB
E + AC
BC
ABC

Thus A, B, C can be estimated unconfounded, while D is confounded with AB, E is confounded with AC (their generators!)

$$\widehat{\text{Effect}} = \frac{\sum_{i=1}^{8} \pm y_i}{4} \Rightarrow \text{Var}(\widehat{\text{Effect}}) = \frac{1}{16} \cdot 8\sigma^2 = \underline{\underline{\frac{\sigma^2}{2}}}.$$

$\sigma^2 = 2 \Rightarrow \operatorname{Var}(\widehat{\text{Effect}}) = 1$

Thus, say that an effect is significant if

$$|\widehat{\text{Effect}}| > z_{0.025}\sqrt{1} = \underline{1.96}$$

Thus from MINITAB-result:
B, C, BC are the only significant effects

A significant $\Leftrightarrow z_{\alpha/2} \leq 1.8750$

$\Leftrightarrow \alpha/2 \geq 0.0304 \Leftrightarrow \alpha \geq 0.0608$

Thus: smallest sign level making A significant is $\underline{\underline{\alpha = 0.0608}}$ ("P-value")

c) We have $m = 7$ estimated effects.
(1) The median of the 7 number is the 4th smallest, which is 1.8750
Preliminary estimate is $1.5 \cdot 1.8750 = 2.8125$

(2) Throw out all effects with abs. value $\geq 2.5 \cdot 2.8125 = 7.033$.
Then 6 remains, with median $\frac{0.2250 + 1.8750}{2} = 1.05$

So $PSE = 1.5 \cdot 1.05 = \underline{\underline{1.575}}$ q.e.d.

OPPGAVE 3

a) Model $Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$

where $\sum_i \alpha_i = \sum_j \beta_j = \sum_i \gamma_{ij} = \sum_j \gamma_{ij} = 0$

$\varepsilon_{ijk} \sim N(0, \sigma^2)$, all independent.

Test statistic for interaction: $F_3 = \frac{S_3^2}{S^2} = \frac{\frac{SS(AB)}{(a-1)(b-1)}}{\frac{SSE}{ab(n-1)}} = \frac{\frac{SS(AB)}{6}}{\frac{SSE}{12}}$

(reject for large values)

Under $H_0$ is $F_3 \sim$ Fisher $(6, 12)$

so critical value is $f_{6,12,0.05} = 3.00$

Observed $F_3$ is 2.29 so we do <u>not</u> reject.

Main effects:

A: $H_0: \alpha_i = 0$ for all $i$ vs. $H_1$: not so

Test statistic:

$$F_1 = \frac{\frac{SS(A)}{a-1}}{\frac{SSE}{ab(n-1)}} = \frac{MSA}{MSE} = 93.52$$

$\sim$ Fisher $(2, 12)$ under $H_0$

Reject at 5% level

(critical value 3.89)

B: $H_0: \beta_j = 0$ for all $j$ vs. $H_1$: not so

$F_2 = 1330.48$ ($\sim$ Fisher $(3, 12)$)

critical value 3.49.)

b) $SSE = \sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y}_{ij\cdot})^2$

$$SS(AB) = n \sum_i \sum_j (\overline{Y}_{ij\cdot} - \overline{Y}_{i\cdot\cdot} - \overline{Y}_{\cdot j\cdot} + \overline{Y}_{\cdot\cdot\cdot})^2$$

$$\frac{SSE}{\sigma^2} = \sum_i \sum_j \left( \underbrace{\frac{\sum_k (Y_{ijk} - \overline{Y}_{ij\cdot})^2}{\sigma^2}}_{\text{This is } \chi^2_{n-1} \text{ by result!}} \right)$$

Also: Have ab terms in $\sum_i \sum_j$,

so $\frac{SSE}{\sigma^2}$ is $\chi^2_{ab(n-1)}$

Here: $ab(n-1) = 12$

Estimator: $S^2 = \frac{SSE}{12} = MSE = 0.537$

estimate

$$\frac{SS(AB)}{\sigma^2} \sim \chi^2_6 \Rightarrow \frac{SSE + SS(AB)}{\sigma^2} \sim \chi^2_{18}$$

$\Rightarrow \frac{SSE + SS(AB)}{18}$ is estimator of $\sigma^2$

$= \frac{8.99 + 6.45}{18} = 0.8578$ (estimate)

## Oppgave 4

For the model (*) we compute

$P(X \in \langle 0, 40]) = F(40) = 0.2212$

$P(X \in \langle 40, 60]) = F(60) - F(40) = 0.2090$

$P(X \in \langle 60, 80]) = F(80) - F(60) = 0.2019$

$P(X \in \langle 80, 100]) = F(100) - F(80) = 0.1583$

$P(X \in \langle 100, 120]) = F(120) - F(100) = 0.1042$

$P(X > 120) = 1 - F(120) = 0.1054$

The expected values are found by multiplying these by 65

Then:

$$\chi^2 = \frac{(7 - 65 \cdot 0.2212)^2}{65 \cdot 0.2212} + \frac{(14 - 65 \cdot 0.2090)^2}{65 \cdot 0,2090}$$

$$+ \cdots \quad = 10.15$$

Reject if $\chi^2 > \chi^2_{0.05,\, 6-1} = 11.070$

so do not reject at 5% (P-value is 0.071)

If parameters not given, we would use the maximum likelihood estimates and perform the same computation.
– but df would then be reduced by 2 (one for each estimated parameter).