5.2.8. *Approksimasjon av* $E[g(X_1,\ldots,X_n)]$ *og* $Var[g(X_1,\ldots,X_n)]$

La $(X_1,\ldots,X_n)$ være en stokastisk vektor, og sett

$$E(X_j) = \mu_j\ ,\quad Var(X_j) = \sigma_j^2\ ,\quad j = 1,\ \ldots, n$$

Betrakt funksjonen $y = g(x_1,\ldots,x_n)$ som antas ha kontinuerlige partiell-deriverte til og med 2. orden i en omegn om $(\mu_1,\ldots,\mu_n)$. Hvis samtlige σ_j^2 er tilstrekkelig små, vil med stor sannsynlighet hver enkelt X_j ligge nær μ_j slik at ledd av formen $c_{jk}(X_j-\mu_j)(X_k-\mu_k)$ blir relativt små. I så fall vil en sannsynligvis ikke gjøre så stor feil om en erstatter

(5.96) $$Y = g(X_1,\ldots,X_n)$$

med

(5.97) $$Z = g(\mu_1,\ldots,\mu_n) + \sum_{j=1}^{n} \frac{\partial g(\mu_1,\ldots,\mu_n)}{\partial \mu_j}(X_j-\mu_j)$$

(En utvikler altså $g(x_1,\ldots,x_n)$ i Taylorrekke omkring punktet $(\mu_1,\ldots,\mu_n)$ og tar bare med førstegradsleddene.)

Propagation of errors
Delta method

Ved å utnytte dette får en følgende formler som kan nyttes ved approksimativ beregning av forventningsverdi og varians for en funksjon $g(X_1,\ldots,X_n)$, som tilfredsstiller betingelsene ovenfor:

(5.98) $$E[g(X_1,\ldots,X_n)] \approx g(\mu_1,\ldots,\mu_n)$$

(5.99) $$Var[g(X_1,\ldots,X_n)] \approx \sum_j \left(\frac{\partial g(\mu_1,\ldots,\mu_n)}{\partial \mu_j}\right)^2 \sigma_j^2 + 2\sum_{j<k} \frac{\partial g}{\partial \mu_j}\frac{\partial g}{\partial \mu_k} Cov(X_j, X_k)$$

Øving 5.11. La (μ_1, μ_2, μ_3) representere fysikalske størrelser som skal måles. På grunn av varierende forsøksbetingelser, målefeil o.l., oppfattes måleresultatene som realisasjoner av en stokastisk vektor (X_1, X_2, X_3) der

$$E(X_j) = \mu_j,\ Var(X_j) = \sigma_j^2 \ \text{ og }\ Cov(X_j,X_k) = \rho_{jk}\sigma_j\sigma_k\ ;\ j,k = 1,2,3$$

Finn et approksimativt uttrykk for forventningsverdi og varians av funksjonen

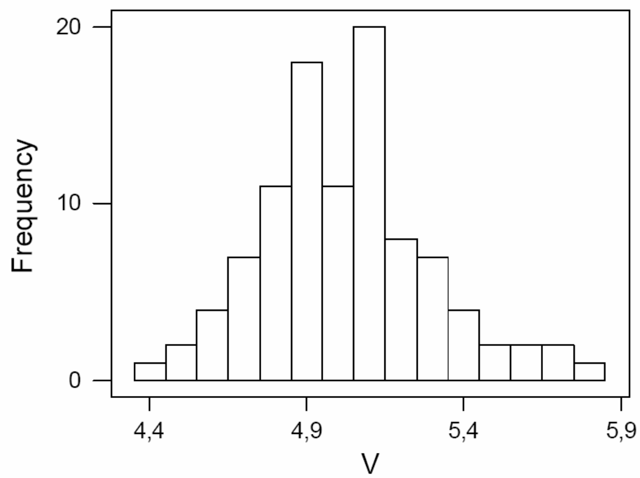
$$Y = kX_1X_2X_3$$

der k er en konstant.

Approksimasjon til forventing og varians for funksjonar av tilfeldige variable.

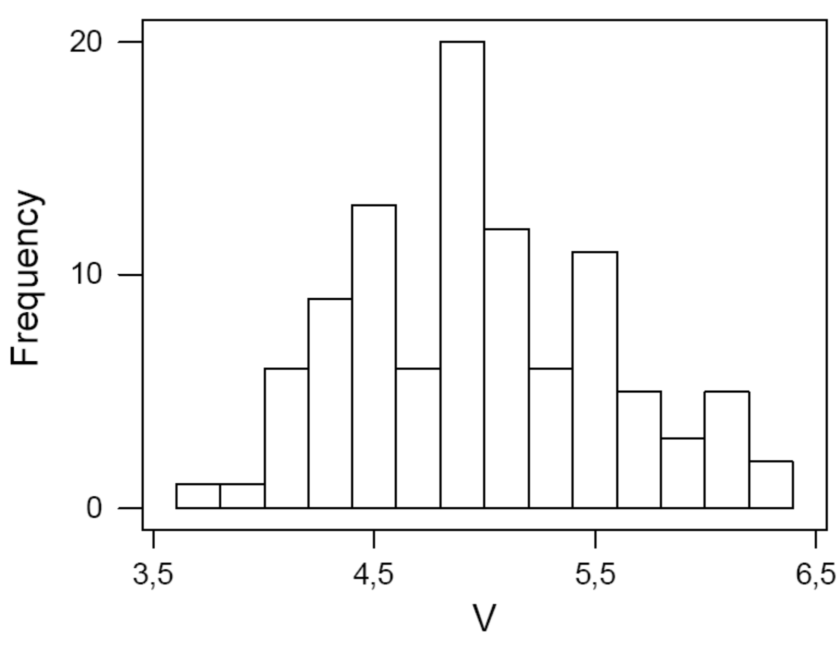
Nedanfor er det synt simuleringsberekningar for estimering av forventing og varians til variabelen V=X/Y der X~N(100,4) og Y~N(20,1). Resultata er basert på 100 simuleringar frå kvar av fordelingane.

$$\hat{\mu}_v = 5.0202$$
$$\hat{\sigma}_v^2 = 0.0712$$

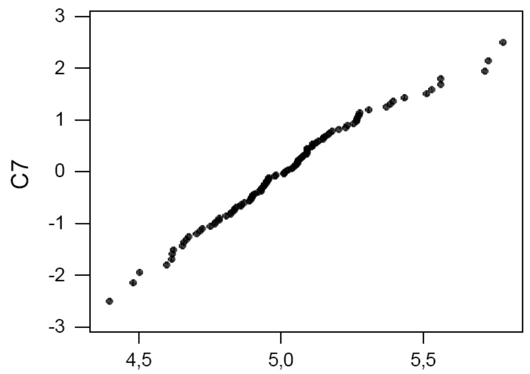


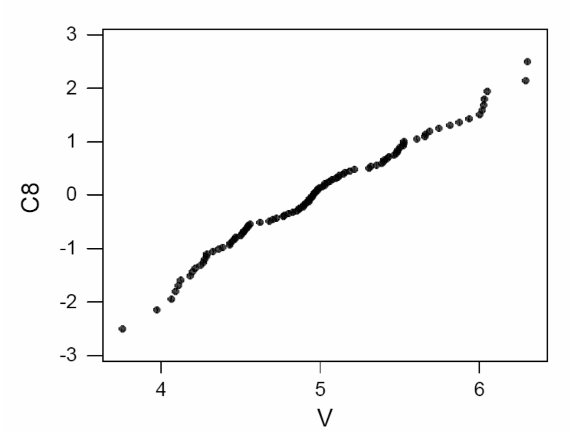
Deretter har ein auka variansen til X til 25 og variansen til Y til 4. Det gav følgjande resultat:

$$\mu_V = 4.9822$$
$$\hat{\sigma}_V^2 = 0.3264$$



Normal plots for each of the two cases





Examples of approximations of expectation and variance

$n = 1$

Suppose $Y = g(X)$, where we know $E(X) = \mu$ and $Var(X) = \sigma^2$.

We want to approximate $E(Y)$ and $Var(Y)$.

Idea: If we know that $Y = aX + b$ for constants a, b, then

$$\begin{aligned} E(Y) &= a\mu + b \\ Var(Y) &= a^2\sigma^2 \end{aligned} \tag{1}$$

We will now approximate Y by Z given by a first order Taylor expansion around $x = \mu$:

$$\begin{aligned} Y &= g(X) \\ Z &= g(\mu) + g'(\mu)(X - \mu) \approx g(X) \end{aligned} \tag{2}$$

This is a good approximation if σ^2 is small, since we then know that X will have a tendency to be near μ.

The approximation of expectation and variance of Y are done by:

$$\begin{aligned} E(Y) &\approx E(Z) = g(\mu) + g'(\mu)(\mu - \mu) = g(\mu) \text{ since } E(X) = \mu \\ Var(Y) &\approx Var(Z) = (g'(\mu))^2\sigma^2 \text{ where we applied (1) to (2)} \end{aligned}$$

$n = 2$

Suppose now $Y = g(X_1, X_2)$, where we know that $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$.

Idea: If we have $Y = a_1X_1 + a_2X_2 + b$, then

$$\begin{aligned} E(Y) &= a_1\mu_1 + a_2\mu_2 + b \\ Var(Y) &= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2Cov(X_1, X_2) \end{aligned} \tag{3}$$

To approximate $E(Y)$ and $Var(Y)$ we first approximate Y by Z given by a first order Taylor expansion of $g(x_1, x_2)$ around $(x_1, x_2) = (\mu_1, \mu_2)$:

$$Z = g(\mu_1, \mu_2) + \frac{\partial g(\mu_1, \mu_2)}{\partial \mu_1}(X_1 - \mu_1) + \frac{\partial g(\mu_1, \mu_2)}{\partial \mu_2}(X_2 - \mu_2) \tag{4}$$

The approximation of expectation and variance of Y are done by:

$$\begin{aligned} E(Y) &\approx E(Z) = g(\mu_1, \mu_2) \\ Var(Y) &\approx Var(Z) = \left(\frac{\partial g(\mu_1, \mu_2)}{\partial \mu_1}\right)^2 \sigma_1^2 + \left(\frac{\partial g(\mu_1, \mu_2)}{\partial \mu_2}\right)^2 \sigma_2^2 + \\ & \quad 2 \cdot \frac{\partial g(\mu_1, \mu_2)}{\partial \mu_1} \cdot \frac{\partial g(\mu_1, \mu_2)}{\partial \mu_1} \cdot Cov(X_1, X_2) \end{aligned}$$

where we used (3) applied to (4).

Note that the last term starting by "2·" can be deleted if X_1, X_2 are independent.

Example

Suppose

$$Y = \frac{X_1}{X_2} = g(X_1, X_2)$$

Then

$$\begin{aligned} g(\mu_1, \mu_2) &= \frac{\mu_1}{\mu_2} \\ \frac{\partial g(\mu_1, \mu_2)}{\partial \mu_1} &= \frac{1}{\mu_2} \\ \frac{\partial g(\mu_1, \mu_2)}{\partial \mu_2} &= -\frac{\mu_1}{\mu_2^2} \end{aligned}$$

so

$$Z = \frac{\mu_1}{\mu_2} + \frac{1}{\mu_2}(X_1 - \mu_1) - \frac{\mu_1}{\mu_2^2}(X_2 - \mu_2)$$

and

$$E(Z) = \frac{\mu_1}{\mu_2}$$

$$Var(Z) = \frac{1}{\mu_2^2}\sigma_1^2 + \frac{\mu_1^2}{\mu_2^4}\sigma_2^2 - \frac{2\mu_1}{\mu_2^3}Cov(X_1, X_2) \quad (5)$$

Numerical examples from slides

In the first example, $X_1 \sim N(100, 2^2),\ X_2 \sim N(20, 1^2)$ are independent.

Then (5) gives:

$$Var(\frac{X_1}{X_2}) \approx \frac{1}{20^2} \cdot 2^2 + \frac{100^2}{20^4} \cdot 1^2 = 0.0725$$

(simulated to 0.0712)

In the second example, $X_1 \sim N(100, 5^2),\ X_2 \sim N(20, 2^2)$ are independent.

Then (5) gives:

$$Var(\frac{X_1}{X_2}) \approx \frac{1}{20^2} \cdot 5^2 + \frac{100^2}{20^4} \cdot 2^2 = 0.3125$$

(simulated to 0.3264)

Note: The simulations are based on drawing 100 pairs of observations (X_1, X_2), and for each pair computing the value of $Y = X_1/X_2$. The empirical variance of the Y, computed from the 100 observations, is then an approximation of the variance that we want to compute. Note that by increasing the number 100, the computed empirical variance will converge to the true value of the variance. This value will probably not be the same as the one we compouted using the Taylor expansion, which is only an approximation.

In the two numerical examples we can compare the numbers computed from the Taylor expansion and the ones obtained by simulation (which in theory are the correct ones, at least if 100 is increased to much larger numbers). It seems that the Taylor approximation is better in the first numerical examples than in the second, and this can be explained by the fact that the variances of X_1 and X_2 smallest in the first example.