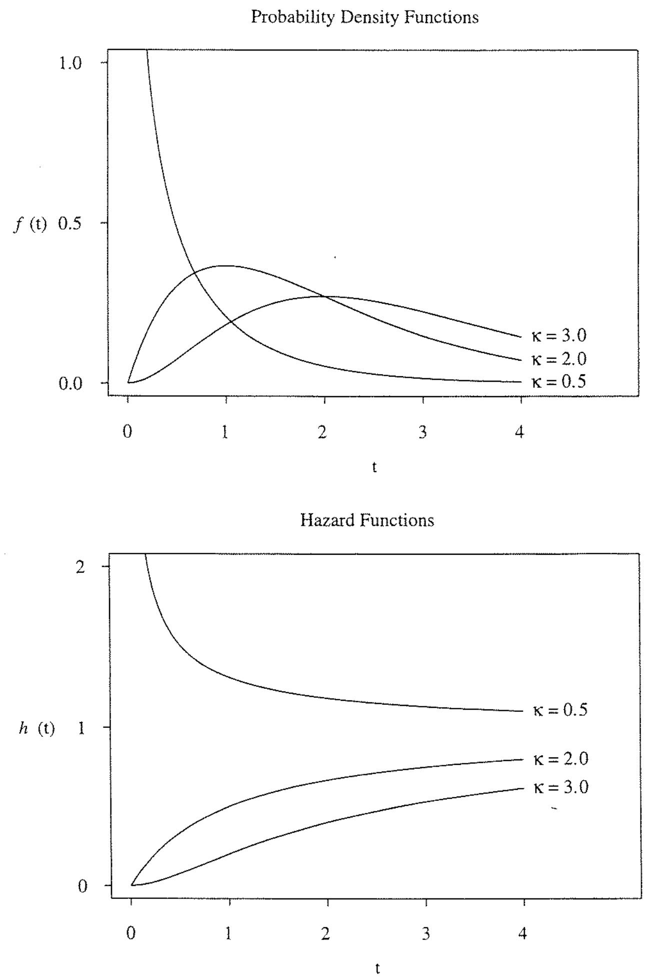


**Figure 1.8.** Gamma p.d.f.'s and hazard functions for $\lambda = 1$ and $k = 0.5, 2.0$, and 3.0.

distributed (i.i.d.) exponential random variables have a gamma distribution. Specifically, if $T_1, \ldots, T_n$ are independent, each with p.d.f. (1.3.2), then $T_1 + \cdots + T_n$ has a gamma distribution with parameters $\lambda$ and $k = n$.

### 1.3.6 Log-Location-Scale Models

A parametric location-scale model for a random variable $Y$ on $(-\infty, \infty)$ is a distribution with p.d.f. of the form



$$f(y) = \frac{1}{b} f_0\left(\frac{y-u}{b}\right) \qquad -\infty < y < \infty, \tag{1.3.18}$$

where $u(-\infty < u < \infty)$ and $b > 0$ are location and scale parameters, and $f_0(z)$ is a specified p.d.f. on $(-\infty, \infty)$. The distribution and survivor functions for $Y$ are $F_0[(y-u)/b]$ and $S_0[(y-u)/b]$, respectively, where

$$F_0(z) = \int_{-\infty}^{z} f_0(w)\,dw = 1 - S_0(z).$$

The standardized random variable $Z = (Y - u)/b$ clearly has p.d.f. and survivor functions $f_0(z)$ and $S_0(z)$, and (1.3.18) with $u = 0, b = 1$ is called the standard form of the distribution.

The lifetime distributions introduced in Sections 1.3.2 to 1.3.4 all have the property that $Y = \log T$ has a location-scale distribution: the Weibull, log-normal, and log-logistic distributions for $T$ correspond to extreme value, normal, and logistic distributions for $Y$. The survivor functions for $Z = (Y - u)/b$ are, respectively,

$$\begin{aligned} S_0(z) &= \exp(-e^z) && \text{extreme value} \\ S_0(z) &= 1 - \Phi(z) && \text{normal} \\ S_0(z) &= (1 + e^z)^{-1} && \text{logistic,} \end{aligned}$$

where $-\infty < z < \infty$ and $\Phi(z)$ is given just before (1.3.11). By the same token, any location-scale model (1.3.18) gives a lifetime distribution through the transformation $T = \exp(Y)$. Note that the survivor function for $T$ can in this case be expressed as

$$\begin{aligned} Pr(T \geq t) &= S_0\left(\frac{\log t - u}{b}\right) \\ &= S_0^*\left[\left(\frac{t}{\alpha}\right)^{\beta}\right], \end{aligned} \tag{1.3.19}$$

where $\alpha = \exp(u)$, $\beta = b^{-1}$, and $S_0^*(x)$ is a survivor function defined on $(0, \infty)$ by the relationship $S_0^*(x) = S_0(\log x)$.

Families of distributions with three or more parameters can be obtained by generalizing (1.3.18) to let $f_0(z)$, $F_0(z)$, or $S_0(z)$ include one or more "shape" parameters. We mention two such families that are useful because they include common two-parameter lifetime distributions as special cases.

The first model is the generalized log-Burr family, for which the standardized variable $(Y - u)/b$ has survivor function of the form

$$S_0(z; k) = \left(1 + \frac{1}{k} e^z\right)^{-k} \qquad -\infty < z < \infty, \tag{1.3.20}$$

where $k > 0$ is a third parameter; it is easily verified that (1.3.20) is a survivor function for all $k > 0$. The special case $k = 1$ gives the standard logistic distribution (see (1.3.14)), and the limit as $k \to \infty$ gives the extreme value distribution (see (1.3.9)). The family of lifetime distributions obtained from (1.3.20) is given by (1.3.19) and has

$$Pr(T \geq t) = \left[1 + \frac{1}{k}\left(\frac{t}{\alpha}\right)^{\beta}\right]^{-k}. \qquad (1.3.21)$$

The log-logistic survivor function is given by $k = 1$, and the Weibull survivor function is given by the limit as $k \to \infty$. Figure 1.9 shows p.d.f.'s for log-Burr distributions (1.3.20) with $k = .5$, 1, 10, and $\infty$. Note that $E(Z)$ and $\text{Var}(Z)$ vary with $k$ (see Problem 1.9) so that the distributions in Figure 1.9 do not have identical means and standard deviations.

Since the generalized log-Burr family includes the log-logistic and Weibull distributions, it allows discrimination between them. It is also a flexible model for fitting to data; inference for it is discussed in Chapter 5.

A second extended model is the generalized log-gamma distribution, which includes the Weibull and log-normal distributions as special cases. The model was originally introduced by specifying that $(T/\alpha)^{\beta}$ has a one-parameter gamma distribution (1.3.17) with index parameter $k > 0$. Equivalently, $W = (Y - u_1)/b_1$, where $Y = \log T$, $u_1 = \log \alpha$ and $b_1 = \beta^{-1}$, has a log-gamma distribution. However, the mean and the variance for the gamma distribution both equal $k$, and as $k$ increases, the gamma and log-gamma distributions do not have limits. The mean and variance for $W$ are (see Problem 1.10) $E(W) = \psi(k)$ and $\text{Var}(W) = \psi'(k)$, where $\psi$ and $\psi'$ are the digamma and trigamma functions (see Appendix B.2). For large $k$ they

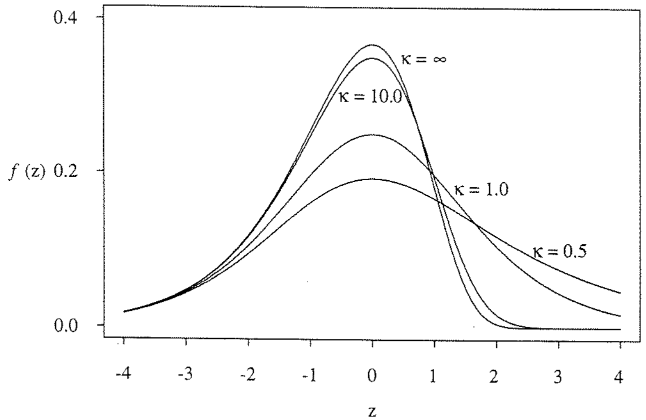


**Figure 1.9.** P.d.f.'s of log-Burr distributions for $k = .5, 1, 10, \infty$.



behave like $\log k$ and $k^{-1}$, respectively (see (B9)), and it is therefore convenient and customary to define a transformed log-gamma variate $Z = k^{1/2}(W - \log k)$, which has p.d.f. (see Problem 1.10)

$$f_0(z; k) = \frac{k^{k-1/2}}{\Gamma(k)} \exp(k^{1/2}z - ke^{k^{-1/2}z}) \qquad -\infty < z < \infty. \qquad (1.3.22)$$

The generalized log-gamma model is then the three-parameter family of distributions for which $Z = (Y - u)/b$ has p.d.f. (1.3.22); the corresponding distribution of $T = \exp(Y)$ is obtained from this, and is called the generalized gamma model. Figure 1.10 shows p.d.f.'s (1.3.22) for $k = .5$, 1, 10, and $\infty$. As for the log-Burr distributions in Figure 1.9, note that $E(Z)$ and $\text{Var}(Z)$ vary with $k$.

For the special case $k = 1$, (1.3.22) becomes the standard extreme value p.d.f. (see (1.3.8)). It can also be shown (see Problem 1.10) that as $k \to \infty$, (1.3.22) converges to the standard normal p.d.f., and thus the generalized gamma model includes the Weibull and log-normal distributions as special cases. The two-parameter gamma distribution (1.3.15) also arises as a special case; in the original $(\alpha, \beta, k)$ parameterization this corresponds to $\beta = 1$, and in the $(u, b, k)$ parameterization with (1.3.22), to $b = k^{-1/2}$. Inference for the generalized gamma and log-gamma distributions is discussed in Chapter 5.

Other extended families may be useful from time to time. For example, one might take $Z = (Y - u)/b$ to have a Student $t$ distribution with $k$ degrees of freedom. Kalbfleisch and Prentice (1980, Sec. 2.2.7) consider a four-parameter model in which $Z$ is a rescaled log $F$ random variable; it includes the generalized log-Burr and log-gamma families as special cases.

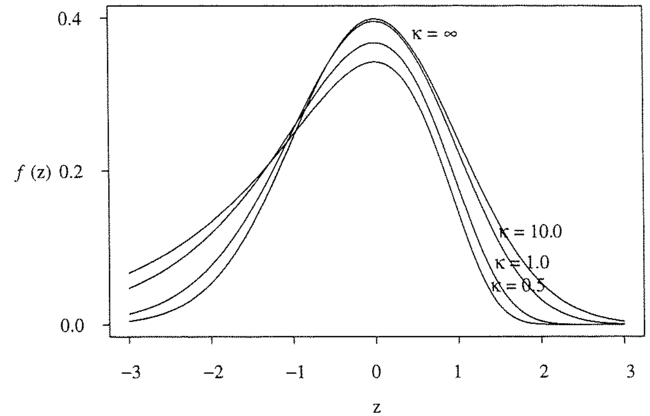


**Figure 1.10.** P.d.f.'s of log-gamma distributions for $k = .5, 1, 10, \infty$.