TMA4275 Lifetime analysis
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About the Exponential Distribution, Poisson Process, Total Time on Test and Barlow-Proschan's Test

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Notation

- $T \sim \text{expon}(\lambda)$ means that T is exponentially distributed with hazard rate λ, i.e. has density
$$f(t) = \lambda e^{-\lambda t} \text{ for } t > 0$$

Properties of the exponential distribution

1. Let $T \sim \text{expon}(\lambda)$. Then
$$P(T > t + s | T > s) = P(T > t)$$
This sais that the distribution of T is "memoryless", i.e. if a unit with lifetime T has reached the age s, the remaining lifetime is still exponentially distributed with parameter λ.
In other words: Let T_s be remaining lifetime for a unit which has reached the age s without failing. Then
$$P(T_s > t) = e^{-\lambda t}$$
i.e. also T_s is $\text{expon}(\lambda)$.
2. Let $T \sim \text{expon}(\lambda)$ and let $W = aT$. Then $W \sim \text{expon}(\lambda/a)$.
3. Let T_i for $i = 1, \ldots, n$ be independent, with $T_i \sim \text{expon}(\lambda_i)$. Let further
$$W = \min(T_1, \ldots, T_n).$$
Then $W \sim \text{expon}(\sum_{i=1}^n \lambda_i)$.
4. In particular if $T_1, \ldots, T_n$ are independent each with distribution $\text{expon}(\lambda)$, then $W \sim \text{expon}(n\lambda)$.
5. Let $T_1, \ldots, T_n$ be independent each with distribution $\text{expon}(\lambda)$. Let the ordering of these be
$$T_{(1)} < T_{(2)} < \cdots < T_{(n)}$$
Then
$$nT_{(1)},\ (n-1)(T_{(2)} - T_{(1)}),\ (n-2)(T_{(3)} - T_{(2)}),\ \ldots,\\ (n-i+1)(T_{(i)} - T_{(i-1)}),\ \ldots, (T_{(n)} - T_{(n-1)})$$

are independent and identically distributed as expon(λ).

This result is given in Theorem D.4 page 584 (Theorem B.4 page 475) in the book. The proof there uses transformations of multidimensional distributions. A more intuitive proof is as follows:

Assume that n units are put on test at time 0. Potential lifetimes of these are $T_1, \ldots, T_n$, and hence

$$T_{(1)} = \min(T_1, \ldots, T_n).$$

From point 4 follows that $T_{(1)} \sim \text{expon}(n\lambda)$, and from this follows by point 2 that $nT_{(1)} \sim \text{expon}(\lambda)$.

After time $T_{(1)}$ there are $n-1$ unfailed units. At time $s = T_{(1)}$ each of these has by point 1 a remaining lifetime which is expon(λ). It follows from this that we from time $T_{(1)}$ and onwards have the same situation as at time 0, only that there are now $n-1$ instead of n units on test. Therefore the time to next failure, $T_{(2)} - T_{(1)}$, is distributed as the minimum of $n-1$ expon(λ) variables and hence is expon($(n-1)\lambda$). Then again by point 2 we get that $(n-1)(T_{(2)} - T_{(1)})$ is expon(λ). That $(n-1)(T_{(2)} - T_{(1)})$ is independent of $nT_{(1)}$ follows from point 1 which says that the distribution of T_s is the same whatever s is.

This reasoning can be continued at time $T_{(2)}$ in an obvious fashion, and we finish by concluding that $T_{(n)} - T_{(n-1)}$ is expon(λ).

6. Let the situation be as in point 5. Total Time on Test (TTT) at the times $T_{(i)}$ are,

$$\begin{aligned}
Y_1 &\equiv \mathcal{T}(T_{(1)}) = nT_{(1)} \\
Y_2 &\equiv \mathcal{T}(T_{(2)}) = nT_{(1)} + (n-1)(T_{(2)} - T_{(1)}) \\
Y_3 &\equiv \mathcal{T}(T_{(3)}) = nT_{(1)} + (n-1)(T_{(2)} - T_{(1)}) + (n-2)(T_{(3)} - T_{(2)}) \\
\vdots &\quad \vdots \\
Y_n &\equiv \mathcal{T}(T_{(n)}) = nT_{(1)} + (n-1)(T_{(2)} - T_{(1)}) + \cdots (T_{(n)} - T_{(n-1)}) \\
&= T_{(1)} + T_{(2)} + \cdots T_{(n)}
\end{aligned}$$

The result of point 5 is that $Y_1, Y_2 - Y_1, \ldots, Y_n - Y_{n-1}$ are i.i.d. expon(λ). But that means that the points $Y_1, Y_2, \ldots, Y_n$ on a single time axis constitute a Poisson process with intensity λ (since the "times" between events in a Poisson process are i.i.d. expon(λ)). This means in turn (by a known result on Poisson processes) that conditionally given $Y_n = y_n$, the $Y_1, \ldots, Y_{n-1}$ will have the same distribution as the ordering of $n-1$ independent variables which are uniform on $(0, y_n)$. (Intuitively this means that if we know the time y_n of the nth event in a Poisson process, then the distribution of the $n-1$ first correspond to $n-1$ independent uniform drawings in the interval $(0, y_n)$).

Dividing by y_n (and putting a capital letter for Y_n), we obtain that under the conditions of point 5, the vector

$$\left(\frac{Y_1}{Y_n}, \frac{Y_2}{Y_n}, \ldots, \frac{Y_{n-1}}{Y_n}\right)$$

has a distribution which corresponds to the ordering of $n-1$ independent uniform variables on $(0,1)$.

This means that Barlow-Proschan's test statistic,

$$W = \frac{Y_1}{Y_n} + \frac{Y_2}{Y_n} + \ldots + \frac{Y_{n-1}}{Y_n}$$

under the null hypothesis of exponentiality has the same distribution as the sum of $n-1$ independent random variables which are uniform on $(0,1)$. Thus

$$E(W) = \frac{n-1}{2}, \quad Var(W) = \frac{n-1}{12}$$

since the expectation and variance of a uniform distribution on $(0,1)$ are, respectively, $1/2$ and $1/12$. Note finally that for n large (presumably will $n \geq 6$ do) is W approximately normally distributed by the central limit theorem. This makes it simple to compute approximate p-values for Barlow-Proschan's test.