TMA4275 Lifetime analysis
Spring 2009
*
The standard confidence interval for positive parameters

Bo Lindqvist

The standard confidence interval

Let $\hat{\theta}$ be the MLE of a (one-dimensional) parameter θ. Then general theory (beyond this course) states that, if the number of observations n is large,

$$\frac{\hat{\theta}-\theta}{\mathrm{SD}(\hat{\theta})} \approx N(0,1) \quad \text{(approximately)}$$

This approximation still holds if we replace $SD(\hat{\theta})$ by an estimate $\widehat{\mathrm{SD}(\hat{\theta})}$, obtained for example by replacing the θ appearing in the expression for $SD(\hat{\theta})$ by $\hat{\theta}$.

It follows that

$$P(-1.96 < \frac{\hat{\theta}-\theta}{\widehat{\mathrm{SD}(\hat{\theta})}} < 1.96) \approx 0.95$$

Rearranging the inequalities within $P(\cdot)$ we get

$$P(\hat{\theta} - 1.96\,\widehat{\mathrm{SD}(\hat{\theta})} < \theta < \hat{\theta} + 1.96\,\widehat{\mathrm{SD}(\hat{\theta})}) \approx 0.95 \tag{1}$$

which defines the *standard 95% confidence interval* for θ:

$$\hat{\theta} \pm 1.96\,\widehat{\mathrm{SD}(\hat{\theta})}.$$

(Of course we may change the 1.96 to obtain other percentages than 95%.)

The standard confidence interval for positive parameters

This method is used by MINITAB to compute confidence intervals for any positive parameter.

We will use the following property of MLE:

- If $\hat{\theta}$ is the MLE of a parameter θ and $g(x)$ is some function, then $g(\hat{\theta})$ is the MLE of $g(\theta)$ (the theorem of substitution).

By the approximate normality of any MLE (see beginning of this note), and the fact that $\ln\hat{\theta}$ must be the MLE of $\ln\theta$ by the just menitoned property of MLE, it follows that

$$\frac{\ln\hat{\theta}-\ln\theta}{\mathrm{SD}(\ln\hat{\theta})} \approx N(0,1) \tag{2}$$

But from Taylor expansion of the natural logarithm we have

$$\ln\hat\theta \approx \ln\theta + \frac{1}{\theta}(\hat\theta - \theta),$$

and therefore,

$$\mathrm{Var}(\ln\hat\theta) \approx \frac{\mathrm{Var}(\hat\theta)}{\theta^2} \quad \text{and hence} \quad \mathrm{SD}(\ln\hat\theta) \approx \frac{SD(\hat\theta)}{\theta}.$$

Substituting this in equation (2) we get

$$\frac{\ln\hat\theta - \ln\theta}{\mathrm{SD}(\hat\theta)/\theta} \approx N(0,1)$$

and by estimating the standard deviation of $\ln\hat\theta$ we get further

$$\frac{\ln\hat\theta - \ln\theta}{\widehat{\mathrm{SD}(\hat\theta)}/\hat\theta} \approx N(0,1)$$

In a similar way as in equation (1) we now get

$$P(\ln\hat\theta - 1.96\,\widehat{\mathrm{SD}(\hat\theta)}/\hat\theta < \ln\theta < \ln\hat\theta + 1.96\,\widehat{\mathrm{SD}(\hat\theta)}/\hat\theta) \approx 0.95.$$

The inequalities inside the $P(\cdot)$ are of course equivalent to the ones we get by taking the exponential functions of all terms, i.e. we have,

$$P(e^{\ln\hat\theta - 1.96\,\widehat{\mathrm{SD}(\hat\theta)}/\hat\theta} < e^{\ln\theta} < e^{\ln\hat\theta + 1.96\,\widehat{\mathrm{SD}(\hat\theta)}/\hat\theta}) \approx 0.95$$

or

$$P(\hat\theta e^{-1.96\,\widehat{\mathrm{SD}(\hat\theta)}/\hat\theta} < \theta < \hat\theta e^{1.96\,\widehat{\mathrm{SD}(\hat\theta)}/\hat\theta}) \approx 0.95.$$

This defines *the standard interval for positive parameters*, which in short be written

$$\hat\theta\; e^{\pm 1.96\,\widehat{\mathrm{SD}(\hat\theta)}/\hat\theta}$$