Problem 1

a) K-M estimator:

\[ R(t) = \sum_{t_i < t} \frac{n_i - d_i}{n_i} \]

<table>
<thead>
<tr>
<th>t_i</th>
<th>n_i</th>
<th>d_i</th>
<th>\frac{n_i - d_i}{n_i}</th>
<th>R(t_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td>10</td>
<td>1</td>
<td>\frac{9}{10}</td>
<td>0.9</td>
</tr>
<tr>
<td>134</td>
<td>8</td>
<td>1</td>
<td>\frac{7}{8}</td>
<td>0.8375</td>
</tr>
<tr>
<td>136</td>
<td>7</td>
<td>1</td>
<td>\frac{6}{7}</td>
<td>0.8571</td>
</tr>
<tr>
<td>170</td>
<td>6</td>
<td>1</td>
<td>\frac{5}{6}</td>
<td>0.8333</td>
</tr>
<tr>
<td>210</td>
<td>4</td>
<td>1</td>
<td>\frac{3}{4}</td>
<td>0.75</td>
</tr>
</tbody>
</table>

\[ R(180) = R(170) = 0.5625 \]

Greenwood's formula:

\[ \text{Var} R(t) = (R(t))^2 \cdot \sum_{t_i < t} \frac{d_i}{n_i - d_i} \]

So

\[ \text{Var} R(180) = (0.5625)^2 \left( \frac{1}{10.9} + \frac{1}{8.7} + \frac{1}{7.6} + \frac{1}{6.5} \right) \]

\[ = 0.12883 \cdot 0.1651^2 \]

So

\[ \text{SD} R(180) = 0.1651 \]

6) Order the times: 27, 124, 136, 170, 191, 210, 308, 441, 511, 559

\[ R(180) = \frac{k}{10} = \frac{6}{10} \]

\[ X \sim \text{bin}(10, R(180)) \]

So \[ \text{Var}(R(180)) = \frac{R(180)(1-R(180))}{10} \]

which is estimated by \[ \text{Var}(R(180)) = \frac{0.6 \cdot 0.4}{10} = 0.1549^2 \]

So \[ \text{SD} \text{ of } 50R(180) = 0.1549 \]
It is seen that the estimates are fairly similar, but $\hat{R}(t)$ is lower, and we have
complete data.

(c) Know that $E(T) = \int_0^\infty R(t) dt$ in general.

So ideally one would use the area under the KM-curve from 0 to $\infty$ to estimate $E(T)$. But this is $\infty$, so it could be reasonable to integrate only to the maximum observed time, here 410 for the censored data.

$$\text{Area} = 27.1 + 97.0.9 + 12 \cdot 0.3825 + 34 \cdot 0.625$$

$$+ 40 \cdot 0.5625 + 200 \cdot 0.4215$$

$$= 253.525 = \hat{E}(T)$$

From complete data: $E(T) = \overline{T} = 262.2$
Problem 2

\( W_j(t) \) = \# failures in \((0, t]\), \( j = 1, 2 \)

which are unbiased estimates for

\[ W_j(t) = E[W_j(t)] \quad \text{for } j = 1, 2. \]

\( W_j(t) = W_j(t) \) so increasing (decreasing) trend
is equivalent to convex (concave) plot.
Thus, seems to be a convexity here, indicating increasing trends.

\( W_j(30) \) is expected \# failures in a month.
For each:

\[ W_j(t) \sim \text{Poisson} \left( W_j(t) \right) \]

so \( W_j(30) \sim \text{Poisson} \left( W_j(30) \right) \)

\[ W_1(30) = 4, \quad \text{SD}(W_1(30)) = \sqrt{4} = 2 \quad \text{since } W_1(t) \sim \text{Poisson}. \]

\[ W_2(30) = 1, \quad \text{SD}(W_2(30)) = \sqrt{1} = 1 \]

b) \( H_0: W_j(t) \) is constant vs \( H_1: W_j(t) \) is increasing.

Test statistic:

\[ Z = 2 \sum_{j=1}^{N_j} \ln \left( \frac{S_j}{w_j(t)} \right) \]

\[ Z = 2 \left[ \ln \left( \frac{30}{1} \right) + \ln \left( \frac{30}{12} \right) + \ln \left( \frac{30}{21} \right) + \ln \left( \frac{30}{28} \right) \right] \approx 3.22 \]
Under $H_0$ is $Z_1 \sim \chi^2_{29,4}$ so reject $H_0$

\[
Z_1 = \frac{Z_1^2}{\chi^2_{0.95, 8}} = 2.73
\]

Thus: Do not reject $H_0$ for Machine 1.

\[
Z_2 = 2 \left[ \ln \frac{30}{8} + \ln \frac{30}{13} + \ln \frac{30}{18} + \ln \frac{30}{21} + \ln \frac{30}{25} + \ln \frac{30}{26} + \ln \frac{30}{29} \right]
\]

= 6.77

Under $H_0$ is $Z_2 \sim \chi^2_{27.14}$ so reject $H_0$

\[
Z_2 < \chi^2_{0.95, 14} = 6.37
\]

Thus: Do not reject $H_0$ for Machine 2.

Pooled Mill 116k test:

$H_0$: both $w_1(t)$ and $w_2(t)$ are constant, but not necessarily equal

$H_1$: at least one of them is increasing.
$$Z = Z_1 + Z_2 = 3.77 + 6.22 = 10.54$$

Under $H_0$, $Z_{permed} \sim \chi^2_{8+14=22}$, so reject $H_0$.

$$Z_{permed} \leq \chi^2_{0.975, 22} = 10.34.$$  

Thus, we reject $H_0$, and conclude that at least one of the $w_j(t)$ is increasing.

---

**Problem 3**

(a) $N_j \sim \text{Poisson}(W_j(0))$; $j = 1, 2$.

Here $W_1(t) = Ct^{1/5} = E(N_1)$

$W_2(t) = 8Ct^{1/5} = E(N_2)$

See that $\delta = \frac{E(N_2)}{E(N_1)}$.

(b) General:

$$L = \prod_{i=1}^{N(2)} \left\{ \frac{N(3)}{W'(S_i)} \right\} e^{-W(S_i)}$$

So

$$\log\text{-likelihood is}$$

$$l = \sum_{c=1}^{N(2)} \ln w(S_i') - \frac{1}{c} \ln W(c)$$
\[ L(d, \beta, S) = \sum_{i=1}^{N_1} \ln (d + \ln \beta + (\beta - 1) \ln S_{i1}) - 1.30^\beta \]

\[ + \sum_{i=1}^{N_2} \left( \ln S + \ln d + \ln \beta + (\beta - 1) \ln S_{i2} \right) - 5.1 \cdot 30^\beta \]

\[ = (N_1 + N_2) \ln d + (N_1 + N_2) \ln \beta \]

\[ + N_2 \ln S + (\beta - 1) \ln - 1(1 + 8) \cdot 30^\beta \]

where \( N = \sum_{i=1}^{N_1} \ln S_{i1} + \sum_{i=1}^{N_2} \ln S_{i2} \)

\[ c) \quad S = \frac{N_2}{N_1} = \frac{11}{7} = 1.75 \]

\[ \beta = \frac{11}{11 \ln 30 - 32.1426} = 2.0871 \]

\[ \gamma = \frac{11}{(1 + 1.73) \cdot 30^{2.0871}} = 0.003305 \]
The parameters are all positive, so we use the standard interval for positive parameters given as (general)

\[ \theta \in \pm 1.96 \frac{\text{S.D.}}{\bar{x}} \]

So for $d$: $0.003305 \times 1.96 \frac{\sqrt{0.528 \times 10^{-5}}}{0.003305} = (4.449 \times 10^{-5}, 0.2456)$

For $\beta$: $2.0871 \pm 1.96 \times 1.156 = (1.156, 3.769)$

For $k$: $(0.5123, 5.978)$

Confidence can be used to test $H_0: \beta = 1$ vs $H_1: \beta \neq 1$ with sign level 5%: reject $H_0$ if $1 \notin$ confidence interval. This is the case, so we reject $H_0$.

$H_0: \beta = 1$ corresponds to $\omega_i(t) = 1, \omega_j(t) = 5$ which is exactly that both $\omega_i(t)$ and $\omega_j(t)$ are constant, but possibly different
d) Test statistic \( W = 2 \left[ \ell(\hat{\lambda}, \hat{S}) - \ell(1, 1) \right] \) is approx \( \chi^2 \) under \( H_0 \), where \( \ell(1, 1) \) are the maximum likelihood estimates of \( \lambda \) and \( S \) when \( \beta = 1 \).

We need to find \( \hat{\lambda}(1), \hat{S}(1) \):

When \( \beta = 1 \) is the log-likelihood

\[
\ell(\lambda, 1, S) = (N_1 + N_2) \ln \lambda + N_2 \ln S - \lambda (1 + S) \cdot 30
\]

\[
\frac{\partial \ell}{\partial \lambda} = \frac{N_1 + N_2}{\lambda} - (1 + S) \cdot 30
\]

\[
\frac{\partial \ell}{\partial S} = \frac{N_2}{S} - 1.30
\]

Solve likelihood equation \( \frac{\partial \ell}{\partial \lambda} = \frac{\partial \ell}{\partial S} = 0 \) gives

\[
\hat{\lambda}(1) = \frac{N_1}{30} = \frac{4}{30} = 0.133
\]

\[
\hat{S}(1) = \frac{N_2}{N_1} = \frac{7}{4} = 1.75
\]

and \( \ell(\hat{\lambda}(1), \hat{S}(1)) = 11 \cdot \ln \left( \frac{4}{30} \right) + 7 \cdot \ln \left( \frac{7}{4} \right) - \frac{4}{30} (1 + \frac{7}{4}) \cdot 30 \)

\[
= -29.24662
\]

So \( W = 2 \left[ \ell(\hat{\lambda}, \hat{S}) - \ell(1, 1) \right] = 4.73 \)
Reject at 5% if \( W \geq 3.84 \), so we reject.