TMA4275 LIFETIME ANALYSIS

Slides 6: Nelson-Aalen estimator and TTT plot

Bo Lindqvist
Department of Mathematical Sciences
Norwegian University of Science and Technology
Trondheim

http://www.math.ntnu.no/∼bo/
bo@math.ntnu.no

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WHY IS AN ESTIMATE OF $Z(t)$ USEFUL?

Note first that $Z'(t) = z(t)$. Thus,

- T is IFR $\Leftrightarrow z(t)$ is *increasing* $\Leftrightarrow Z(t)$ is *convex*
- T is DFR $\Leftrightarrow z(t)$ is *decreasing* $\Leftrightarrow Z(t)$ is *concave*

Thus a plot of an estimate $\hat{Z}(t)$ can give us information on whether the distribution of T is IFR (*increasing failure rate*) or DFR (*decreasing failure rate*).

Recall that $R(t) = e^{-Z(t)}$, so

$$Z(t) = -\ln R(t)$$

Thus - if $\hat{R}_{KM}(t)$ is the KM-estimator for $R(t)$, then we can define,

$$\begin{aligned}\hat{Z}_{KM}(t) &= -\ln \hat{R}_{KM}(t) \\ &= -\ln \prod_{T_{(i)} \leq t} \frac{n_i - d_i}{n_i} \\ &= -\sum_{T_{(i)} \leq t} \ln \left(1 - \frac{d_i}{n_i}\right) \\ &\approx \sum_{T_{(i)} \leq t} \frac{d_i}{n_i}\end{aligned}$$

where we used that for small x is

$$-\ln(1 - x) \approx x$$

THE NELSON-AALEN ESTIMATOR FOR $Z(t)$

The Nelson-Aalen estimator (NA-estimator) is simply defined by

$$\hat{Z}_{NA}(t) = \sum_{T_{(i)} \leq t} \frac{d_i}{n_i}$$

It can then be shown that its variance can be estimated by

$$\widehat{Var(\hat{Z}_{NA}(t))} = \sum_{T_{(i)} \leq t} \frac{d_i}{n_i^2}$$

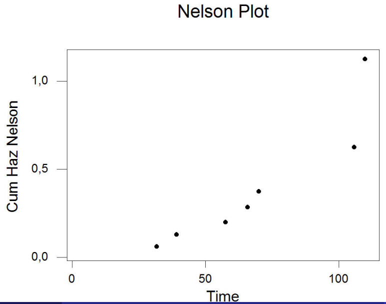
Note: The Nelson-Aalen estimator is *not* included in MINITAB (only "hazard plot" which is in fact not a correct). For this course has been made a *MINITAB Macro* (see MINITAB Macros on the Software webpage).

In the following we shall have a closer look at how the Nelson-Aalen estimator can be motivated from properties of the exponential distribution.

EXAMPLE: NELSON-AALEN ESTIMATOR

Row	C1	C2
1	31,7	1
2	39,2	1
3	57,5	1
4	65,0	0
5	65,8	1
6	70,0	1
7	75,0	0
8	75,2	0
9	87,5	0
10	88,3	0
11	94,2	0
12	101,7	0
13	105,8	1
14	109,2	0
15	110,0	1
16	130,0	0

Row	Time	Numb at risk	1/Numb at risk	Cum Haz Nelson	Survival Nelson
1	31,7	16	0,062500	0,06250	0,939413
2	39,2	15	0,066667	0,12917	0,878827
3	57,5	14	0,071429	0,20060	0,818244
4	65,8	12	0,083333	0,28393	0,752820
5	70,0	11	0,090909	0,37484	0,687401
6	105,8	4	0,250000	0,62484	0,535348
7	110,0	2	0,500000	1,12484	0,324705



Suppose an item with lifetime T is still alive at time s. The probability of surviving an additional t time is then

$$\begin{aligned} R(t \mid s) &\equiv P(T > s + t \mid T > s) \\ &= \frac{P(T > s + t \cap T > s)}{P(T > s)} \\ &= \frac{R(s+t)}{R(s)} \end{aligned}$$

This is called the *conditional survival function* of the item at time t, or *the distribution of the residual life*. The following is its expectation, called *Mean Residual Life:*

$$\begin{aligned} MRL(t) &= \int_0^\infty R(t \mid s)dt = \int_0^\infty \frac{R(s+t)}{R(s)}dt \\ &= \frac{1}{R(s)} \int_s^\infty R(t)dt \end{aligned}$$

PROPERTIES OF THE EXPONENTIAL DISTRIBUTION:
1. The memoryless property

Write $T \sim \text{expon}(\lambda)$ if $f(t) = \lambda e^{-\lambda t}$; $R(t) = P(T > t) = e^{-\lambda t}$, $t > 0$.

For $T \sim \text{expon}(\lambda)$ we therefore have

$$R(t \mid s) = P(T > s + t \mid T > s) = \frac{R(s+t)}{R(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = R(t).$$

This is called *the memoryless property* of the exponential distribution.

For any age s, the remaining life has same distribution as for a new item.

2. Let $T \sim \text{expon}(\lambda)$ and let $W = aT$. Then $W \sim \text{expon}(\lambda/a)$.

   Proof:

$$P(W > w) = P(aT > w) = P(T > \frac{w}{a}) = e^{-(\frac{\lambda}{a})w}$$

3. Let T_i for $i = 1, \ldots, n$ be independent, with $T_i \sim \text{expon}(\lambda_i)$. Let $W = \min(T_1, \ldots, T_n)$.. Then $W \sim \text{expon}(\sum_{i=1}^{n} \lambda_i)$.

   Proof:

$$\begin{aligned} P(W > w) &= P(\min(T_1, \cdots, T_n) > w) \\ &= P(T_1 > w, T_2 > w, \cdots, T_n > w) \\ &= P(T_1 > w)P(T_2 > w) \cdots P(T_n > w) \\ &= e^{-(\lambda_1 + \cdots + \lambda_n)w}, \end{aligned}$$

   so $W \sim \text{expon}(\lambda_1 + \cdots + \lambda_n)$

4. In particular if $T_1, \ldots, T_n$ are independent each with distribution expon(λ), then

$$W = \min(T_1, \ldots, T_n) \sim \text{expon}(n\lambda)$$

So a series system of n components with lifetimes that are independent and exponentially distributed with hazard rate λ, has a lifetime which is exponenital with hazard rate $n\lambda$ and hence

$$\text{MTTF} = \frac{1}{n\lambda} = \frac{\text{Component MTTF}}{n}$$

5. Let $T_1, \ldots, T_n$ be independent each with distribution $\text{expon}(\lambda)$. Let the ordering of these be

$$T_{(1)} < T_{(2)} < \cdots < T_{(n)}$$

Then

$$\begin{gathered} nT_{(1)} \\ (n-1)(T_{(2)} - T_{(1)}) \\ (n-2)(T_{(3)} - T_{(2)}) \\ \vdots \\ (n-i+1)(T_{(i)} - T_{(i-1)}) \\ \vdots \\ (T_{(n)} - T_{(n-1)}) \end{gathered}$$

are independent and identically distributed as $\text{expon}(\lambda)$.

5b. Let $T_1, \ldots, T_n$ be independent each with distribution expon(λ). Let the ordering of these be

$$T_{(1)} < T_{(2)} < \cdots < T_{(n)}$$

Then

$$\begin{aligned}
T_{(1)} &\sim \text{expon}(n\lambda) \\
T_{(2)} - T_{(1)} &\sim \text{expon}((n-1)\lambda) \\
T_{(3)} - T_{(2)} &\sim \text{expon}((n-2)\lambda) \\
&\vdots \\
T_{(i)} - T_{(i-1)} &\sim \text{expon}((n-i+1)\lambda) \\
&\vdots \\
T_{(n)} - T_{(n-1)} &\sim \text{expon}(\lambda)
\end{aligned}$$

are independent with the displayed exponential distributions.

PROOF OF THE EQUIVALENT CLAIMS OF 5 AND 5b

To go from 5b to 5, we use property 2 of the exponential distribution. Thus we prove only 5b here.

Assume that n units are put on test at time 0. Potential lifetimes of these are $T_1, \ldots, T_n$, and hence $T_{(1)} = \min(T_1, \ldots, T_n)$, so by property 4 above we already have $T_{(1)} \sim \text{expon}(n\lambda)$.

After time $T_{(1)}$ there are $n-1$ unfailed units. At time $s = T_{(1)}$ each of these has by property 1 a remaining lifetime which is expon(λ). It follows from this that we from time $T_{(1)}$ and onwards have the same situation as at time 0, only that there are now $n-1$ instead of n units on test. Therefore the time to next failure, $T_{(2)} - T_{(1)}$, is distributed as the minimum of $n-1$ expon(λ) variables and hence is expon$((n-1)\lambda)$. That $T_{(2)} - T_{(1)}$ is independent of $T_{(1)}$ follows from property 1 which says that, for the exponential distribution, the distribution of the remaining lifetime is the same whatever be the age of the item.
This reasoning can be continued at time $T_{(2)}$ in an obvious fashion, and we finish by concluding that $T_{(n)} - T_{(n-1)}$ is expon(λ).