# TMA4275 LIFETIME ANALYSIS

## Slides 7: Introduction to parametric inference in lifetime models

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- *Parametric distributions*
- *Types of censoring*
- *Representation of censored data*
- *Likelihood construction for arbitrary censored data*

# PARAMETRIC DISTRIBUTIONS IN MINITAB

- Smallest extreme (Gumbel)
- Weibull
- 3-parameter Weibull
- Exponential
- 2-parameter exponential
- Normal
- Lognormal
- 3-parameter Lognormal
- Logistic
- Log logistic
- 3-parameter Loglogistic

A model for a lifetime $T$ is called *parametric* if it is given on the form $f(t;\theta)$ , $F(t;\theta)$, etc., for functions which are "fixed" except for a parameter value $\theta$ which is allowed to vary in some prespecified interval or area. *Note:* $\theta$ may be a vector of several parameters.

*Examples:*

- $f(t;\theta) = \frac{1}{\theta} e^{-t/\theta}$, $F(t;\theta) = 1 - e^{-t/\theta}$; $\theta > 0$
- $f(t;\alpha,\theta) = \frac{1}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1} e^{-(t/\theta)^{\alpha}}$, $F(t;\alpha,\theta) = 1 - e^{-(t/\theta)^{\alpha}}$; $\alpha > 0, \theta > 0$

**Aim:** *To estimate or test hypotheses about the true* $\theta$ *in a sample (possibly censored) of observations of* $T$.

On the other hand, a model for a lifetime $T$ is called *nonparametric* if it allows any shape of the functions $f$ and $F$.

# WHY PARAMETRIC MODELS?

- Complements nonparametric techniques.
- Parametric models can be described concisely with just a few parameters, instead of having to report an entire curve.
- It is possible to use a parametric model to extrapolate (in time) to the lower or upper tail of a distribution.
- Parametric models provide smooth estimates of failure time distributions. In practice it is often useful to compare various parametric and nonparametric analyses of a data set.

Lifetime data typically include *censored* data, meaning that:

- some lifetimes are known to have occurred only within certain intervals.
- The remaining lifetimes are known exactly.

*Categories of censoring:*

- right censoring
- left censoring
- interval censoring

*Right censoring is the most common way of censoring.* Different subtypes of right censoring can be considered. A common way of presenting right-censored data is as follows:

$n$ units are observed, with potential i.i.d. lifetimes $T_1, T_2, \cdots, T_n$. For each $i$, we observe a time $Y_i$ which is either the true lifetime $T_i$, or a censoring time $C_i < T_i$, in which case the true lifetime is "to the right" of the observed time $C_i$.

The observation from a unit is the pair $(Y_i, \delta_i)$ where the *censoring indicator* $\delta_i$ is defined by

$$\delta_i = \begin{cases} 1 & \text{if} \quad Y_i = T_i \\ 0 & \text{if} \quad Y_i = C_i, \text{ in which case it is known that } T_i > Y_i \end{cases}$$

The lifetime $T_i$ for the $i$th individual is *left censored* if it is *less than* a censoring time $C_i$, that is, the event of interest has already occurred for the individual before that person is observed in the study at time $C_i$.

In this case we *observe* the pair $(Y_i, \epsilon_i)$, where

$$\begin{aligned} Y_i &= \max(T_i, C_i) \\ \epsilon_i &= \begin{cases} 1 & \text{if} \quad T_i \geq C_i \\ 0 & \text{if} \quad T_i < C_i \end{cases} \end{aligned}$$

.

# EXAMPLE OF LEFT CENSORING

(From *Klein and Moeschberger).* In a study to determine the distribution of the time until first marijuana use among high school boys in California, the question was asked, "When did you first use marijuana?" One of the responses was "I have used it but can not recall just when the first time was." A boy who chose this response is indicating that the event had occurred prior to the boy's age at interview but the exact age at which he started using marijuana is unknown. This is an example of a left-censored event time.

*If, on the other hand, the answer was "I never used it". What kind of censoring would this correspond to?*

When the lifetime is only known to occur *within an interval*, the lifetime is said to be *interval censored.*

*Example:* Suppose that patients in a clinical trial have *periodic follow-up* and the patient's event time $T_i$ is only known to fall in an interval $(L_i, R_i]$.

Interval censoring may also occur in industrial experiments where there is periodic inspection for proper functioning of equipment items.

Assume we have data for $n$ units with *potential lifetimes* $T_1, T_2, \cdots, T_n \sim_{i.i.d.} f(t;\theta)$.

Noncensored unit: Record the failure time $T_i$ (*ideal case*)

Censored unit: Exact lifetime $T_i$ is not recorded; all we know is that $T_i \in [a, b]$ for an interval of times.

Here

- $a$ is the observed time, and $b = \infty$ for *right censorings*
- $a = 0$, while $b$ is the observed time for *left censorings*
- $0 < a < b < \infty$ for an interval censoring between the observed interval limits $a$ and $b$

*Data for censored data are entered as follows:*

| Unit no | start variable | end variable | Frequency (optional) |
|---|---|---|---|
| 1 | $a_1$ | $b_1$ | $f_1$ |
| 2 | $a_2$ | $b_2$ | $f_2$ |
| 3 | $a_3$ | $b_3$ | $f_3$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

*NOTE:* An uncensored observation is entered by letting both $a_i$ and $b_i$ equal the observed lifetime.

| Obs. type | Lower bound $a_i$ | Upper bound $b_i$ | Likelihood contribution |
|---|---|---|---|
| Exact lifetime | 1.7 | 1.7 | $f(1.7;\theta)$ |
| Right cens. | 2.0 | $\infty$ | $1-F(2.0;\theta)$ |
| Left cens. | 0 | 0.5 | $F(0.5;\theta)$ |
| Interval cens. | 1.0 | 1.5 | $F(1.5;\theta)-F(1.0;\theta)$ |

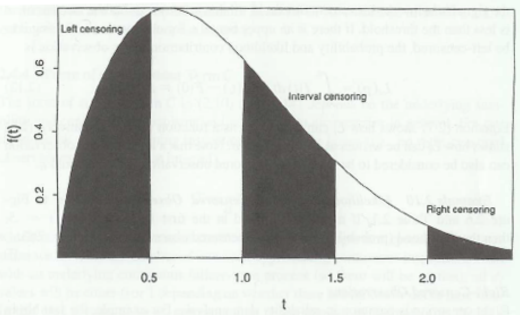


Figure 2.6. Likelihood contributions for different kinds of censoring.

With general censoring, each observed time $T_i$ is represented in our data as $\{T_i \in [a_i, b_i]\}$.

The likelihood function is then defined as

$$\begin{aligned} L(\theta) &= \text{Probability of gettting the observed data under parameter } \theta \\ &= P_\theta(T_1 \in [a_1, b_1] \cap \cdots \cap T_n \in [a_n, b_n]) \\ &= P_\theta(T_1 \in [a_1, b_1]) \cdots P_\theta(T_n \in [a_1, b_1]) \\ &= (F(b_1; \theta) - F(a_1; \theta)) \cdots (F(b_n; \theta) - F(a_n; \theta)) \\ &= \prod_{i=1}^{n} (F(b_i; \theta) - F(a_i; \theta)) \end{aligned}$$

Generally, the information $T_i \in [a_i, b_i]$ contributes by $F(b_i;\theta) - F(a_i;\theta)$.

**Special cases:**

- *Right censoring:* Here $b_i = \infty$, so the contribution to likelihood function is

$$F(\infty;\theta) - F(a_i;\theta) = 1 - F(a_i;\theta) = R(a_i,\theta)$$

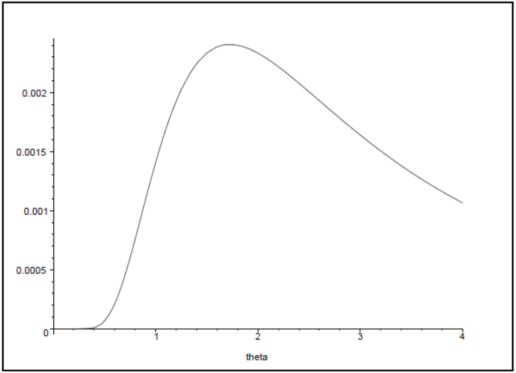
- *Left censoring:* Here $a_i = 0$, so contribution to likelihood is

$$F(b_i;\theta) - F(0;\theta) = F(b_i,\theta)$$

- *Interval censoring:* Contribution is $F(b_i;\theta) - F(a_i;\theta)$
- *Exact observed lifetime:* Then $a_i = b_i$. Write instead $b_i = a_i + h$, so contribution is $F(a_i + h;\theta) - F(a_i;\theta) \approx f(a_i)h$. So we let the contribution be be just $f(a_i)$.

LIKELIHOOD FOR MODEL $f(t;\theta) = (1/\theta)e^{-t/\theta}$

$$L(\theta) = (\frac{1}{\theta}e^{-1.7/\theta}) \cdot (e^{-2.0/\theta}) \cdot (1 - e^{-0.5/\theta}) \cdot (e^{-1.0/\theta} - e^{-1.5/\theta})$$




Maximum likelihood estimate: $\hat{\theta} = 1.725$

**Distribution Analysis, Start = A and End = B**

Variable Start: A  End: B

| Censoring Information | Count |
|---|---|
| Uncensored value | 1 |
| Right censored value | 1 |
| Interval censored value | 1 |
| Left censored value | 1 |

Estimation Method: Maximum Likelihood

Distribution: Exponential

Parameter Estimates

| | | Standard | 95,0% Normal CI | |
|---|---|---|---|---|
| Parameter | Estimate | Error | Lower | Upper |
| Mean | 1,72529 | 0,998421 | 0,554978 | 5,36353 |

Log-Likelihood = -6,029

Goodness-of-Fit
Anderson-Darling (adjusted) = 4,933

Characteristics of Distribution

| | | Standard | 95,0% Normal CI | |
|---|---|---|---|---|
| | Estimate | Error | Lower | Upper |
| Mean(MTTF) | 1,72529 | 0,998421 | 0,554978 | 5,36353 |
| Standard Deviation | 1,72529 | 0,998421 | 0,554978 | 5,36353 |
| Median | 1,19588 | 0,692053 | 0,384682 | 3,71771 |
| First Quartile(Q1) | 0,496336 | 0,287228 | 0,159657 | 1,54299 |
| Third Quartile(Q3) | 2,39177 | 1,38411 | 0,769363 | 7,43543 |
| Interquartile Range(IQR) | 1,89543 | 1,09688 | 0,609706 | 5,89244 |

# LIKELIHOOD FOR RIGHT CENSORED DATA

*Recall notation:* The observations are $(y_i, \delta_i)$ for $i = 1, 2, \ldots, n$, where the *censoring indicator* $\delta_i$ is defined by

$$\delta_i = \begin{cases} 1 & \text{if} \quad y_i \text{ is the lifetime } t_i \\ 0 & \text{if} \quad y_i \text{ is a censoring time, so } t_i > y_i \end{cases}$$

Then the likelihood function becomes

$$L(\theta) = \prod_{i:\delta_i=1} f(y_i;\theta) \cdot \prod_{i:\delta_i=0} R(y_i;\theta)$$