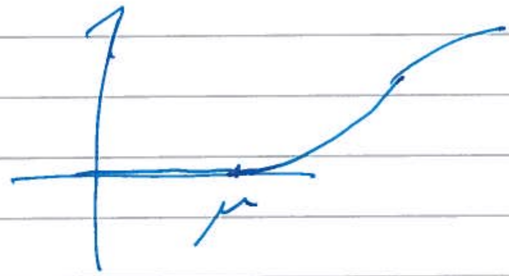
2010

Problem 2

$$F(t;\alpha,\mu) = e^{-\frac{\alpha}{t-\mu}} \qquad \text{for } t>\mu,\ \alpha>0.$$

a) Cum. CDF ?

- Increasing in t, yes
- $F(0;\alpha,\mu)=0$? yes
- $F(\infty;\alpha,\mu)=1$?



$$\cancel{\lim_{t\to}}\ \lim_{t\to\infty} e^{-\frac{\alpha}{t-\mu}} = 1 \ ?$$

$$\frac{\alpha}{t-\mu} \to 0 \ \text{ as } t\to\infty, \text{ thus OK.}$$

$$z(t) = \frac{f(t)}{1-F(t)}$$

$$f(t) = e^{-\frac{\alpha}{t-\mu}} \cdot \frac{\alpha}{(t-\mu)^2} = \frac{\alpha}{(t-\mu)^2} e^{-\frac{\alpha}{t-\mu}} \quad \text{for } t>\mu.$$

$$z(t) = \frac{\frac{\alpha}{(t-\mu)^2} e^{-\frac{\alpha}{t-\mu}}}{1-e^{-\frac{\alpha}{t-\mu}}} = \frac{\alpha}{(t-\mu)^2} \cdot \frac{1}{e^{\frac{\alpha}{t-\mu}}-1}.$$

$$F(t_p) = p$$

$$\Updownarrow$$

$$e^{-\frac{\alpha}{t-\mu}} = p$$

$$\Updownarrow$$

$$\frac{\alpha}{t-\mu} = -\ln p$$

$$t-\mu = \frac{\alpha}{-\ln p}$$

$$t_p = \mu + \frac{\alpha}{-\ln p}$$

c) $\cancel{l(\alpha,\mu)} = \cancel{\sum_{i=1}^{n} \frac{\alpha}{(t_i-\mu)^2} e^{-\sum_{i=1}^{n} \frac{\alpha}{t_i-\mu}}}$

$$l(\alpha,\mu) = \sum_{i=1}^{n} \log f(t_i;\alpha,\mu)$$

$$\cancel{= \alpha \sum_{i=1}}$$

$$= \sum_{i=1}^{n} \left[\ln \alpha - 2\ln(t_i-\mu) - \frac{\alpha}{t_i-\mu}\right]$$

$$= n\ln\alpha - 2\sum_{i=1}^{n} \ln(t_i-\mu) - \alpha \sum_{i=1}^{n} \frac{1}{t_i-\mu}$$

d) $$\cancel{\frac{\partial \ell}{\partial \alpha}}\ \ell'(\alpha) = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{1}{t_i - \mu} = 0$$

$$\cancel{\sum_{i=1}^{n} \frac{1}{t_i-\mu} =}$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \frac{1}{t_i - \mu}}$$

$$\cancel{\frac{\partial^2 \ell}{\partial \alpha^2}}\ \ell''(\alpha) = -\frac{n}{\alpha^2}$$

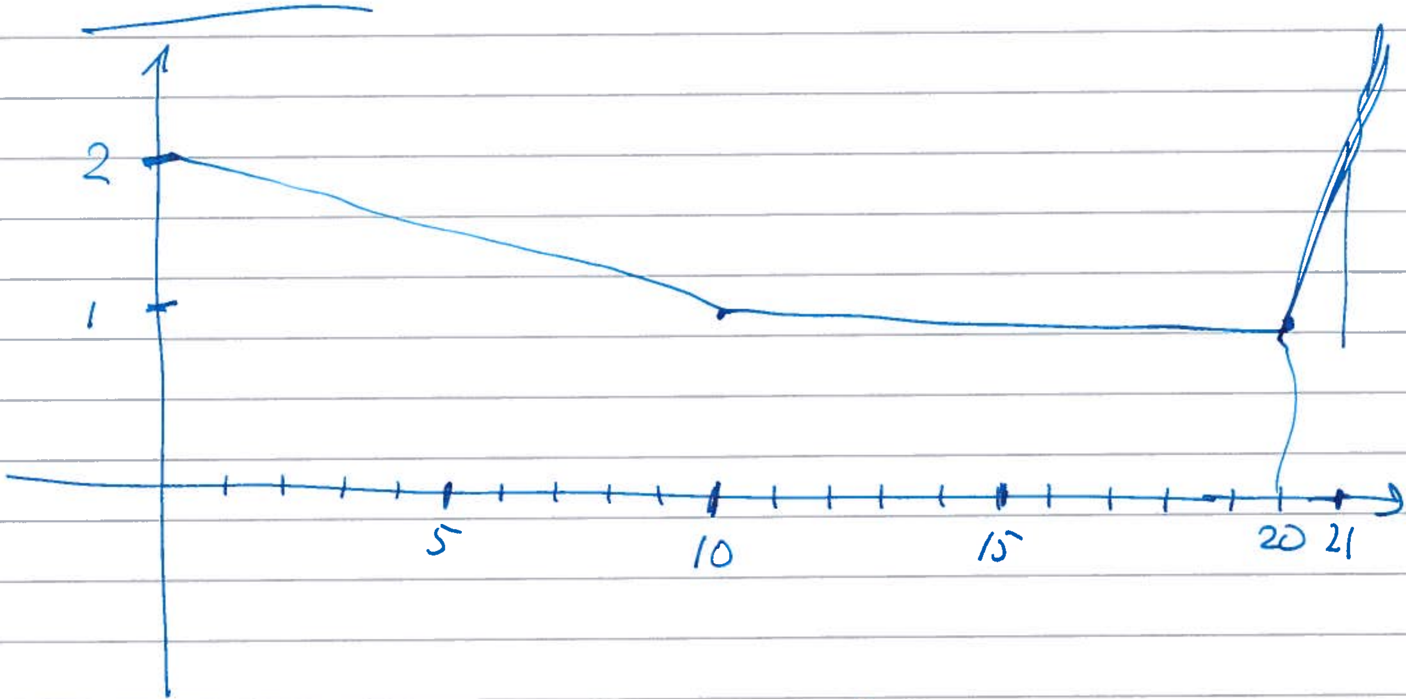
~~So Var(α̂)~~ $$\widehat{Var(\hat{\alpha})} \approx \frac{1}{-\ell''(\hat{\alpha})} = \frac{\hat{\alpha}^2}{n}$$

e) $$\left(-\log F(t)\right)^{-1} = \left(\frac{\alpha}{t-\mu}\right)^{-1} = \frac{t-\mu}{\alpha}$$

So: the points $\left(y_i, -\log(1-\hat{R}_{KM}(y_i))\right)$

are on the line "$y = \frac{1}{\alpha} t - \frac{\mu}{\alpha}$"

Problem 3



a) $N(t)$ is an NHPP with intensity $z(t)$.

ROCOF: $\overset{w(t)}{=} z(t)$

NHPP: $N(0) = 0$

~~NBh~~ $N(s,t)$ are indep when intervals are disjoint, are Poisson($\int_s^t z(u)\,du$)

$= W(t) - W(s)$

b) $W(t) = \begin{cases} \text{For } 0 < t < 10: \\ \int_0^t \left(2 - \frac{1}{10}u\right) du \end{cases}$

$= 2t - \frac{1}{20} t^2$ for $0 < t < 10$

$W(10) = 2 \cdot 10 - \frac{1}{20} \cdot 10^2 = 20 - \frac{100}{20} = \underline{\underline{15}}$

For $10 < t < 20$:

$$W(t) = 15 + (t-10) = \underline{\underline{5+t}}$$

$$W(20) = \underline{25}$$

For $t > 20$: ~~25~~

$$W(t) = 25 + \int_{20}^{t} (u - 19)\, du$$

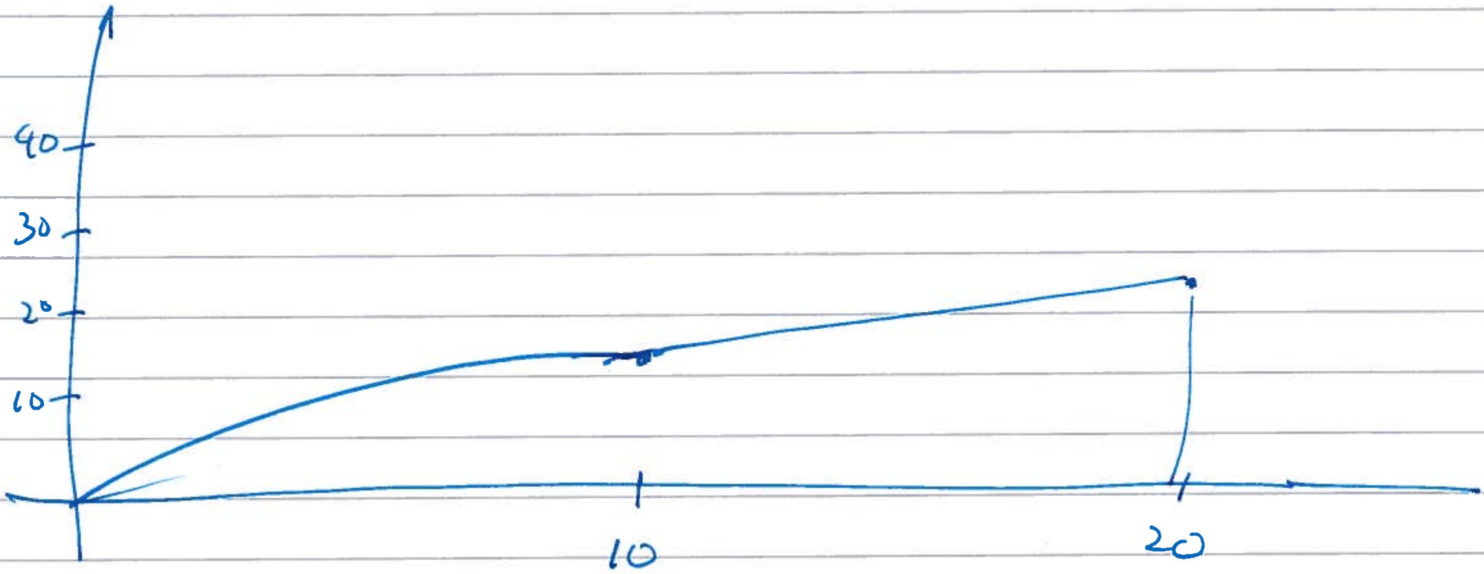
$$= 25 + \Big|_{20}^{t} \left(\tfrac{1}{2}u^2 - 19u\right)$$

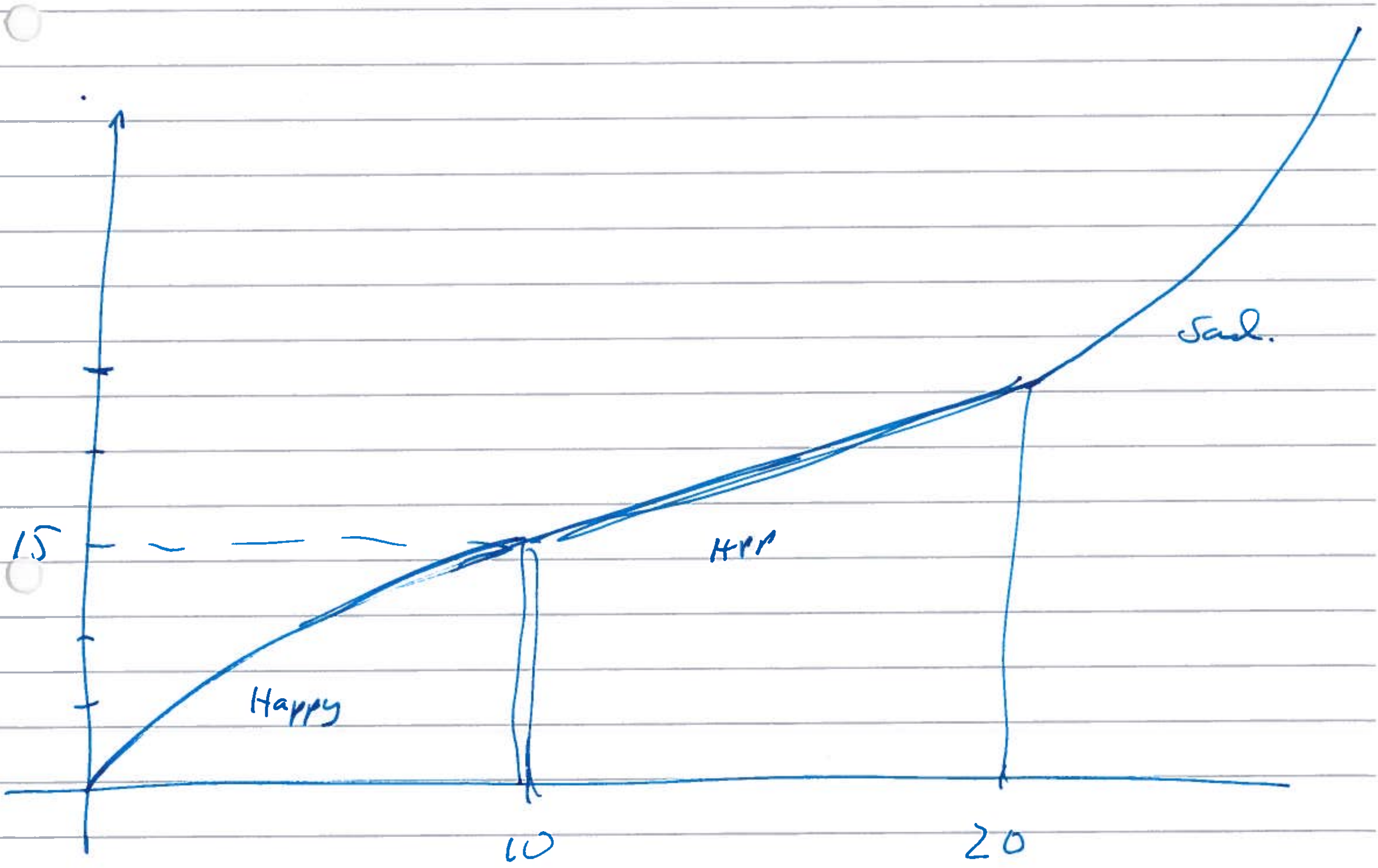
$$= 25 + \tfrac{1}{2}t^2 - 19t - \tfrac{1}{2}\cdot 20^2 + 19\cdot 20$$

$$= 25 + \tfrac{1}{2}t^2 - 19t - 200 + 380$$

$$= \underline{205 + \tfrac{1}{2}t^2 - 19t}$$ ~~405~~

$$W(20) = 205 + \tfrac{1}{2}\cdot 20^2 - 19\cdot 20$$
$$= 205 + 200 - 380 = \underline{\underline{25}}$$





$$E\,N(12) = W(12) = 5 + 12 = \underline{\underline{17}}$$

c) $P(N(2) = 1) =$

Poisson$(W(2))$ i.e. Poisson$\left(2 \cdot 2 - \frac{1}{20} \cdot 2^2\right)$

$$= 4 - \frac{4}{20} = 4 - \frac{1}{5} = \frac{20}{5} - \frac{1}{5} = \frac{19}{5}$$

$P(X=x) = \frac{\lambda^x}{x!} e^{-\lambda}$

So $P(N(2)=1) = \cancel{\frac{19}{5}}\, \frac{19}{5} \cdot e^{-\frac{19}{5}} = \underline{0.085}$

d) $P(T_1 > t) = P(\text{no failures in } (0,t))$

$= P(N(0,t) = 0) = \underline{e^{-W(t)}}$

$\Rightarrow$

$P(\text{first failure not in day } 1)$

$= P(T_1 > 1) = e^{-W(1)}$

$= e^{-(2 - \frac{1}{20})} = e^{-\frac{39}{20}} = \underline{0.142}$

~~P(~~

What happens in time ~~(1,3)~~ interval $(1,3)$ is independent on what happens before time 1.

So this is just

$P(N(1,3) = 0) =$

Poisson $(W(3) - W(1)) = 2 \cdot 3 - \frac{1}{20} \cdot 3^2 - 2 + \frac{1}{20}$

$= 4 - \frac{8}{20} = 4 - \frac{2}{5} = \frac{18}{5}$

So $P(N(1,3)=0) = e^{-\frac{18}{5}} = \underline{0,0273}$

$$\cancel{\lambda\left(2 - \frac{1}{10} + 2 - \frac{3}{10}\right)\cdot 2}$$

$$\cancel{= \left(4 - \frac{4}{10}\right) 2 = 8 - \frac{8}{10}}$$

$$\cancel{= 8 - 4}$$

$$\cancel{4 - \frac{4}{10} = \frac{36}{10}} = \frac{18}{5}$$