TMA 4295 Statistical Inference 2014
Homework 3

Problem 1

Reade example 3.6.3 in the book. Thereafter, do exercise 3.47. (Hint: Instead of multiplying by x/t, which is done in the example, you should try to multiply by t^r/x^r for a suitable r).

Problem 2

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli-distributed with parameter p (see the book p. 621), (i.e., $X_1, \ldots, X_n$ are results of n binomial trials). Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

a) Explain why the central limit theorem (CLT) gives that

$$\frac{\sqrt{n}(\bar{X}_n - p)}{\sqrt{p(1-p)}} \xrightarrow{d} N(0,1)$$

b) Explain why

$$\frac{\sqrt{n}(\bar{X}_n - p)}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} \xrightarrow{d} N(0,1)$$

c) Go through the example from the lectures where it is shown that

$$\sqrt{n}(2\arcsin\sqrt{\bar{X}_n} - 2\arcsin\sqrt{p}) \xrightarrow{d} N(0,1)$$

d) How would you use the results in a)-c) to make (approximate) confidence intervals for p?

Problem 3

Let $U \sim N(0,1)$, $V \sim \chi^2_p$, where U and V are independent.

a) Write down the joint density of U, V.

b) Let

$$T = \frac{U}{\sqrt{\frac{V}{p}}}$$

What is the distribution of T (following the elementary course in statistics)?

Find the joint distribution of T, W, where $W = V$, (see p. 223 in the book).

c) Derive the density of the t-distribution with p degrees of freedom.

Problem 4

In this problem you will go through the prrof of Theorem 5.3.1 on p. 218 in the book.

For part **a** the idea is as follows:

1. Explain why it is enough to prove the result when $\mu = 0$ and $\sigma = 1$ (which will simplify the writing furhter in the proof). You must here show by a precise argument that the result for general (μ, σ) can be derived from that of $\mu = 0, \sigma = 1$.

2. Show that S^2 can be expressed alone as a function of the $n - 1$ variables

$$Y_2 = X_2 - \bar{X}, \ldots, Y_n = X_n - \bar{X}$$

3. Write down the joint distribution of $X_1, \ldots, X_n$ (remember the assumption $N(0, 1)$). Define $Y_1 = \bar{X}$ and let $Y_2, \ldots, Y_n$ be as in the previous point. Find the joint distribution $f(y_1, \ldots, y_n)$ of $Y_1, \ldots, Y_n$ by using the multivariate transformation formula (with Jacobi-determinant etc.) Conclude from f that Y_1 is independent of $Y_2, \ldots, Y_n$. Why does it follow from this that $\bar{X}$ and S^2 are independent?

Finally, go through the induction proof of part **c** in the theorem, i.e. the proof that $(n - 1)S^2/\sigma^2$ is chi-square distributed.