TMA 4295 Statistical Inference 2014
Homework 9

Problem 1

Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ (both parameters unknown) and assume that we want to test

$$H_0 : \sigma^2 = \sigma_0^2 \text{ vs } H_1 : \sigma^2 \neq \sigma_0^2$$

The likelihood ratio $\lambda(\boldsymbol{x})$ was derived in the lecture on October 9, to be

$$\lambda(\boldsymbol{X}) = \left(\frac{Z}{n}\right)^{\frac{n}{2}} e^{(n-Z)/2}$$

where

$$Z = \frac{n\hat{\sigma}^2}{\sigma_0^2} = \sum_{i=1}^{n}(X_i - \bar{X})^2/\sigma_0^2,$$

which is χ^2_{n-1} under H_0.

(You should go through this calculation before proceeding!)

It follows that

$$-2\ln\lambda(\boldsymbol{X}) = Z - n - n\ln\frac{Z}{n} \equiv g(Z)$$

Thus, the likelihood ratio test (LRT), which is to reject H_0 if $\lambda(\boldsymbol{X}) \leq c$ is equivalent to rejecting if $g(Z) \geq -2\log c$.

By sketching the curve $g(z)$ it was concluded in the lecture on October 23 that the LRT with significance level α is equivalent to rejecting if

$$Z \leq z_0 \text{ or } Z \geq z_1$$

where z_0, z_1 are determined so that

(1) $g(z_0) = g(z_1)$

(2) $P(Z \leq z_0) + P(Z \geq z_1) = \alpha$

(Go through this, too, before proceeding!)

(a) Assume $n = 10$ and $\alpha = 0.05$. Find z_0 and z_1 by "trial and error".

(This requires calculation of cumulative probabilities of the chi-square distribution. You may for example use R).

(b) If you use the result that $-2\ln\lambda(\boldsymbol{X})$ is approximately chi-square with 1 degree of freedom, you will instead solve the equations

$$g(z_0) = g(z_1) = 3.84$$

Why? Compare with the result of the previous question.

(c) In this question you are asked to show directly that $-2 \ln \lambda(\boldsymbol{X})$ is approximately χ_1^2 under H_0, which you used in the previous question.

The point of departure is that it can be shown (follows in fact from the first problem of the trial exam) that Z is approximately normal for large n, in the sense that

$$\frac{Z - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{d} N(0,1)$$

Explain why we then have

$$U_n \equiv \frac{Z - n}{\sqrt{2n}} \xrightarrow{d} N(0,1)$$

Express $-2 \ln \lambda(\boldsymbol{X})$ by U_n. Show, by using that for y close to 0 is $\ln(1 + y) \approx y - \frac{1}{2}y^2$, that $-2 \ln \lambda(\boldsymbol{X}) \approx U_n^2$. Why does this give the wanted result?

Problem 2

Let X be Poisson(α) and Y be Poisson(β). To find out whether X, Y are identically distributed we will test

$$H_0 : \alpha = \beta \text{ vs } H_1 : \alpha \neq \beta$$

a) Write down the likelihood function $L(\alpha, \beta | x, y)$. Find the MLE for α, β in the full model and under H_0.

b) Show that the likelihood ration can be written

$$\lambda(x, y) = \left(\frac{x+y}{2x}\right)^x \left(\frac{x+y}{2y}\right)^y$$