Modelling of dependence between critical failure and preventive maintenance: The repair alert model

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Abstract

We consider the competing risks problem for a repairable unit which at each sojourn may be subject to either a critical failure, or a preventive maintenance (PM) action, where the latter will prevent the failure. It is reasonable to expect a dependence between the failure mechanism and the PM regime. The paper presents a new model, called the repair alert model, for handling such cases. This model is a special case of random signs censoring, which was introduced by Roger Cooke [Statistics and Probability Letters, 18:307-312, 1993]. The pleasant feature of random signs censoring is that the marginal distribution of the failure time is identifiable. The repair alert model introduces the so called repair alert function, which characterizes the "alertness" of the maintenance crew, and which is shown to be uniquely identifiable from field data. Statistical estimation is considered both nonparametrically and parametrically.

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1 Introduction

Consider a component which is subject to failure at a random time X . Assume that the failure can be avoided by a possible preventive maintenance (PM) at some random time Z . If $Z < X$, then we will not observe the failure but rather the PM event. On the other hand, if $X < Z$, then the failure is experienced. The situation is thus a case of competing risks, where in effect we observe only the pair (Y, δ) , where $Y = \min(X, Z)$ and $\delta = I(Z < X)$ is the indicator of the event $\{Z < X\}$.

During operation, a (competent) maintenance crew is likely to have some information regarding the state of the component. The crew will use this insight to perform maintenance in order to avoid typically more costly component failures. Hence it is reasonable to expect a dependence between the time X of failure of the component and the time Z of PM actions. We are therefore faced with *dependent competing risks*. Cooke (1993, 1996) introduced the notion of *random signs censoring* which is tailored for such cases. In our notation, random signs censoring can be defined as follows:

Definition 1 *Let (X, Z) be a pair of life variables. Then Z is called a random signs censoring of X if the event $\{Z < X\}$ is stochastically independent of X .*

Thus, random signs censoring means that the event that the failure of the component is preceded by PM, is not influenced by the time X at which the component fails or would have failed without PM.

It is well known (Tsiatis, 1975) that the marginal distributions of X and Z are not identifiable from the observed (Y, δ) without making assumptions on the dependence between them. Cooke's random signs censoring makes such assumptions, leading to identifiability of the distribution of X . On the other hand, the distribution of Z is not identifiable in general under random signs censoring. However, it is usually the marginal distribution of X which is of most interest, since it is natural to use this distribution as basis for maintenance optimization.

In the present paper we suggest a new model, the repair alert model, for which Z is a random signs censoring of X , but where we impose additional structure. The idea is to define a so called repair alert function, which describes the “alertness” of the maintenance crew as a function of time. The repair alert model generalizes the model of Langseth and Lindqvist (2003), who used the failure rate function itself as the measure of alertness.

The main result of the paper (Theorem 3) states that whenever there exists a model satisfying the random signs requirements, there is a unique repair alert model having the same distribution of the observable (Y, δ) .

The plan of the paper is as follows. Section 2 lists the basic notation used in the preceding sections, while Section 3 gives a review of random signs censoring. The definition and some basic results of the repair alert model are given in Section 4. Further investigations on its implications are given in Section 5. Section 6 gives a brief treatment of statistical estimation in the repair alert model, using a real data set. Finally, some concluding remarks are given in Section 7.

2 Notation

Throughout the paper we assume that (X, Z) is a pair of continuously distributed life variables, with the property that $P(X = Z) = 0$. We shall use the following notation which appears to be standard.

We let $F_X(t) = P(X \leq t)$ and $F_Z(t) = P(Z \leq t)$ be the cumulative distribution functions of X and Z , respectively. The *subdistribution functions* of X and Z are defined as, respectively, $F_X^*(t) = P(X \leq t, X < Z)$ and $F_Z^*(t) = P(Z \leq t, Z < X)$.

Note that the functions F_X^* and F_Z^* are nondecreasing with $F_X^*(0) = 0$ and $F_Z^*(0) = 0$. Moreover, we have $F_X^*(\infty) + F_Z^*(\infty) = 1$. Any pair of functions K_1, K_2 satisfying these conditions, will later be referred to as a *subdistribution pair*.

We will also use the notion of *conditional distribution functions*, defined by $\tilde{F}_X(t) = P(X \leq t | X < Z)$ and $\tilde{F}_Z(t) = P(Z \leq t | Z < X)$. Note then that $\tilde{F}_X(t) = F_X^*(t)/F_X^*(\infty)$, $\tilde{F}_Z(t) = F_Z^*(t)/F_Z^*(\infty)$.

For convenience we shall throughout the paper assume the existence of corresponding densities of the functions defined above, i.e. $f_X(t) = F_X'(t)$, $f_X^*(t) = F_X^{*'}(t)$, $\tilde{f}_X(t) = \tilde{F}_X'(t)$, and similarly for Z .

3 Random signs censoring

It was mentioned in the introduction that the marginal distribution of X is identifiable under random signs censoring. This follows directly from the definition, since we have

$$\tilde{F}_X(t) = P(X \leq t | X < Z) = P(X \leq t) = F_X(t)$$

Hence the marginal distribution of X is in fact the same as the distribution of the observed occurrences of X .

The following theorem states that a random signs distribution for (X, Z) exists if and only if the conditional distribution function of X is below that of Z .

Theorem 1 (*Cooke, 1993*) *Let K_1, K_2 be a subdistribution pair. Then the following are equivalent*

(i) *There exists a pair (X, Z) of life variables such that Z is a random signs censoring of X , and such that*

$$F_X^*(t) = K_1(t) \text{ for all } t \geq 0, F_Z^*(t) = K_2(t) \text{ for all } t \geq 0$$

(ii) $\frac{K_1(t)}{K_1(\infty)} < \frac{K_2(t)}{K_2(\infty)}$ for all $t > 0$

The intuitive implication of this result is that the condition $\tilde{F}_X(t) < \tilde{F}_Z(t)$ for all $t > 0$ (which corresponds to (ii)), is consistent with Z being a random signs censoring of X .

On the other hand, if $\tilde{F}_X(t) \geq \tilde{F}_Z(t)$ for some t , then there is no joint distribution of (X, Z) for which the random signs requirement holds. For more discussion on random signs censoring and its applications we refer to Cooke (1993, 1996), and Bedford and Cooke (2001, Ch. 9).

4 The repair alert model

Definition 2 *The pair (X, Z) of life variables satisfies the requirements of the repair alert model provided the following two conditions both hold:*

(i) *Z is a random signs censoring of X*

(ii) *There exists an increasing function G with $G(0) = 0$ such that for all $x > 0$,*

$$P(Z \leq z | Z < X, X = x) = \frac{G(z)}{G(x)}, \quad 0 < z \leq x$$

The function G is called *the cumulative repair alert function*. Its derivative g (which we shall assume exists) is called *the repair alert function*.

The repair alert model is hence a specialization of random signs censoring, obtained by introducing the repair alert function G .

Part (ii) of the definition means that, given that there would be a failure at time $X = x$, and given that the maintenance crew will perform a PM before that time (i.e. given that $Z < X$), the conditional density of the time Z of this PM is proportional to the repair alert function g .

The repair alert function is meant to reflect the reaction of the maintenance crew. More precisely, $g(t)$ ought to be high at times t for which failures are expected and the alert therefore should be high. As mentioned above, Langseth and Lindqvist (2003) simply put $g(t) = \lambda(t)$ where $\lambda(t)$ is the failure rate of the marginal distribution of X . The companion paper to the present paper, Langseth and Lindqvist (2004), presents ways to test whether $g(t)$ can be assumed equal to the hazard function $\lambda(t)$. This property of $g(t)$ of course simplifies analyses since it reduces the number of parameters, but at the same time it seems fairly reasonable given a competent maintenance crew.

It will follow from the construction in Theorem 3 that the repair alert model is completely determined by the marginal distribution function F_X , the (cumulative) repair alert function G , the probability $q \equiv P(Z < X)$, and the assumption that X is independent of the event $\{Z < X\}$ (i.e. random signs censoring).

The next theorem shows how the subdistribution- and conditional distribution functions can be expressed by the parameters of the model.

Theorem 2 *Suppose that the pair (X, Z) has a repair alert model defined by $F_X(x)$, $G(t)$ and q . Then*

$$\tilde{F}_X(t) = F_X(t) \tag{1}$$

$$F_X^*(t) = (1 - q)F_X(t) \tag{2}$$

$$\tilde{F}_Z(t) = F_X(t) + G(t) \int_t^\infty \frac{f_X(y)}{G(y)} dy \tag{3}$$

$$\tilde{f}_Z(t) = g(t) \int_t^\infty \frac{f_X(y)}{G(y)} dy \tag{4}$$

$$F_Z^*(t) = q\tilde{F}_Z(t) \tag{5}$$

Proof: Equation (1) is a direct consequence of random signs censoring, while (2) and (5) are trivially satisfied by definition. For (3) we have

$$\begin{aligned} \tilde{F}_Z(t) &= P(Z \leq t | Z < X) \\ &= \int_0^\infty P(Z \leq t | Z < X, X = x) P(x \leq X \leq x + dx | Z < X) \end{aligned}$$

$$= \int_0^\infty \min\left(\frac{G(t)}{G(x)}, 1\right) \cdot f_X(x) dx = F_X(t) + G(t) \int_t^\infty \frac{f_X(x)}{G(x)} dx$$

Finally (4) follows by differentiation.

The main content of the next theorem is that if condition (ii) of Theorem 1 holds, then there is a unique repair alert model having the given set of subdistribution functions.

Theorem 3 *Let K_1, K_2 be a subdistribution pair such that K_2 is differentiable. Then the following are equivalent*

(i) *There exists a pair (X, Z) of life variables which satisfy the requirements of the repair alert model and which are such that*

$$F_X^*(t) = K_1(t) \text{ for all } t \geq 0, \quad F_Z^*(t) = K_2(t) \text{ for all } t \geq 0$$

$$(ii) \quad \frac{K_1(t)}{K_1(\infty)} < \frac{K_2(t)}{K_2(\infty)} \text{ for all } t > 0$$

Moreover, if condition (ii) holds and (X, Z) has the repair alert model in (i), then the cumulative repair alert function G is uniquely (modulo a multiplicative constant) given by

$$G(t) = \exp \left\{ \int_{t_0}^t \frac{\tilde{f}_Z(w)}{\tilde{F}_Z(w) - F_X(w)} dw \right\} \quad (6)$$

for all $t > 0$, where $t_0 > 0$ is a fixed, arbitrary constant.

Proof: That (i) implies (ii) follows at once from Theorem 1. Thus, assume that (ii) holds. Define for convenience,

$$\tilde{K}_i(t) = \frac{K_i(t)}{K_i(\infty)} \text{ for all } t > 0, \quad i = 1, 2$$

Let further $q = K_2(\infty)$ in the following.

Let X, U be a pair of stochastically independent random variables such that $F_X(t) = \tilde{K}_1(t)$ for all t , and $P(U = 1) = 1 - P(U = 0) = q$. We now define the random variable Z conditionally on X, U :

Given $X = x, U = 0$ we put $Z = x + 1$ with probability 1. The interesting part of the construction of the distribution of Z is when $U = 1$. For that case

we introduce the function

$$G(t) = \exp \left\{ \int_{t_0}^t \frac{\tilde{k}_2(w)}{\tilde{K}_2(w) - \tilde{K}_1(w)} dw \right\} \quad (7)$$

for all $t > 0$, where $t_0 > 0$ is a fixed, arbitrary constant. Here $\tilde{k}_2(w) = \tilde{K}_2'(w)$. Then we define the conditional distribution of Z given $X = x, U = 1$ by

$$P(Z \leq z | X = x, U = 1) = \frac{G(z)}{G(x)}, \quad \text{for } 0 < z \leq x$$

It follows by construction that the pair X, Z satisfies the requirements of a repair alert model. Note in particular that $\{U = 1\}$ is equivalent to $\{Z < X\}$. Moreover in the model for (X, Z) we have $F_X^*(x) \equiv P(X \leq x, X < Z) = P(X \leq x, U = 0) = P(X \leq x)P(U = 0) = K_1(x)$ since $K_1(\infty) = P(U = 0)$.

It remains to show that $F_Z^*(z) = K_2(z)$, or equivalently, that $\tilde{F}_Z(z) = \tilde{K}_2(z)$ for all z .

Using the fact that we have constructed a repair alert distribution for X and Z , with given cumulative repair alert function G from (7), we get by combining (3) and (4),

$$\frac{g(t)}{G(t)} = \frac{\tilde{f}_Z(t)}{\tilde{F}_Z(t) - F_X(t)} \quad (8)$$

On the other hand, differentiating (7) and noting that $\tilde{K}_1(t) = F_X(t)$ we get

$$\frac{g(t)}{G(t)} = \frac{\tilde{k}_2(t)}{\tilde{K}_2(t) - F_X(t)}$$

This means that both $y = \tilde{K}_2(t)$ and $y = \tilde{F}_Z(t)$ satisfy the differential equation

$$\frac{g(t)}{G(t)} = \frac{y'}{y - F_X(t)}$$

By reordering, this can be written as a first order linear differential equation, which with the obvious initial condition $y(0) = 0$ must have a unique solution guaranteed by standard theory. But then $\tilde{F}_Z(t) = \tilde{K}_2(t)$ for all t .

The final uniqueness result for G in the theorem follows since G (and g) necessarily must satisfy (8). This equation can be written as a homogeneous

linear differential equation which determines G modulo a constant, with a possible solution given by (6). This completes the proof.

5 Some properties of the repair alert model

In this section we discuss some implications of the repair alert model, in particular how the parameters q and G influence the observed performance of PM and failures. In order to help intuition, we sometimes consider the power version $G(t) = t^\beta$ where $\beta > 0$ is a parameter. Then $g(t) = \beta t^{\beta-1}$ so $\beta = 1$ means a constant repair alert function, while $\beta < 1$ and $\beta > 1$ correspond to, respectively, a decreasing and increasing repair alert function.

Under the random signs assumption, the parameter $q = P(Z < X)$ is connected to the ability to discover “signals” regarding a possibly approaching failure. More precisely, q is understood as the probability that a failure is avoided by a preceding PM.

Given that there will be a PM, one should ideally have the time of PM immediately before the failure. We shall see that this last issue is connected to the function G . Figure 1 shows the conditional distribution function $G(z)/G(x)$ of Z given $X = x, Z < X$, for $G(t) = t^\beta$ and varying β . It is seen that large values of β correspond to distributions with most of its mass near x .

Moreover, it follows from equation (3) that

$$E(Z|Z < X) = \int_0^\infty (1 - \tilde{F}_Z(z))dz = E(X) - E\left[\frac{M(X)}{G(X)}\right]$$

where $M(x) = \int_0^x G(t)dt$. For the special case when $G(t) = t^\beta$, we obtain the simple result

$$E(Z|Z < X) = \frac{\beta}{\beta + 1}E(X) \tag{9}$$

which clearly indicates that good PM performance corresponds to large values of β .

Instead of merely considering the conditional expectation $E(Z|Z < X)$ one may more generally study the conditional *distribution* of Z given $Z < X$. Part (i) of Theorem 4 below gives a result on the behaviour of Z in relation to X for different repair alert functions.

Fig. 1. Graphs of the conditional distribution function $G(z)/G(x)$ of Z given $X = x, Z < X$, for $G(t) = t^\beta$. The diagonal corresponds to $\beta = 1$, the other curves are (from upper to lower) for $\beta = 1/10, 1/5, 5, 10$.

An alternative way of considering the relation between Z and X is via the distribution of the remaining time to the potential failure, given that a PM is performed at time $Z = z$. More specifically, this is the conditional distribution of $X - Z$ given $Z = z, Z < X$. Intuitively, a good PM performance would mean that this distribution is small (stochastically). Part (ii) of Theorem 4 gives a result on the behaviour of this distribution for different repair alert functions.

The density of the conditional distribution of $X - Z$ given $Z = z, Z < X$ may be obtained as follows. Indeed, we know from the definition of the repair alert model that the conditional density of Z given $X = x, Z < X$ is $g(z)/G(x)$, and that the conditional density of X given $Z < X$ is $f_X(x)$. Hence from Bayes' formula we find that the conditional density of $W = X - Z$ given $Z = z, Z < X$ is proportional to $f_X(z + w)/G(z + w)$.

Theorem 4 Suppose (X, Z) has a repair alert distribution with fixed parameters q and F_X , while the cumulative repair alert function is either $G^{(1)}$ or $G^{(2)}$. Suppose further that $G^{(1)}(t)/G^{(2)}(t)$ is an increasing function of t . Then

- (i) The conditional distribution of Z given $Z < X$ under $G^{(1)}$ stochastically dominates the corresponding distribution under $G^{(2)}$.
- (ii) The conditional distribution of $W = X - Z$ given $Z = z, Z < X$ under $G^{(1)}$ is stochastically dominated by the corresponding distribution under $G^{(2)}$.

Proof: For (i), recall first that the conditional distribution of Z given $Z < X$ is given by $\tilde{F}_Z$ in equation (3). Let $\tilde{F}_Z^{(i)}$ be distributions under $G^{(i)}$ ($i = 1, 2$)

and let $\rho(t) = G^{(1)}(t)/G^{(2)}(t)$. Then for all t ,

$$\begin{aligned}\tilde{F}_Z^{(1)}(t) &= F_X(t) + G^{(1)}(t) \int_t^\infty \frac{f_X(y)}{G^{(1)}(y)} dy \\ &= F_X(t) + \rho(t) G^{(2)}(t) \int_t^\infty \frac{f_X(y)}{\rho(y) G^{(2)}(y)} dy \\ &\leq F_X(t) + G^{(2)}(t) \int_t^\infty \frac{f_X(y)}{G^{(2)}(y)} dy = \tilde{F}_Z^{(2)}(t)\end{aligned}$$

since $\rho(y) \geq \rho(t)$ for all $y \geq t$.

Next we prove (ii). Let $P^{(i)}$ be probabilities under $G^{(i)}$ ($i = 1, 2$). Then it follows from the discussion before the proposition that

$$\begin{aligned}P^{(1)}(W \leq w | Z = z, Z < X) &= \int_z^{z+w} \frac{f_X(y)}{G^{(1)}(y)} dy / \int_z^\infty \frac{f_X(y)}{G^{(1)}(y)} dy \\ &= \left(1 + \int_{z+w}^\infty \frac{f_X(y)}{G^{(1)}(y)} dy / \int_z^{z+w} \frac{f_X(y)}{G^{(1)}(y)} dy \right)^{-1} \\ &= \left(1 + \int_{z+w}^\infty \frac{f_X(y)}{\rho(y) G^{(2)}(y)} dy / \int_z^{z+w} \frac{f_X(y)}{\rho(y) G^{(2)}(y)} dy \right)^{-1} \\ &\geq \left(1 + \int_{z+w}^\infty \frac{f_X(y)}{\rho(z+w) G^{(2)}(y)} dy / \int_z^{z+w} \frac{f_X(y)}{\rho(z+w) G^{(2)}(y)} dy \right)^{-1} \\ &= P^{(2)}(W \leq w | Z = z, Z < X)\end{aligned}$$

where we used the fact that $\rho(t)$ is increasing in t .

Theorem 4 immediately implies that if $G(t) = t^\beta$, then (i) the conditional distribution of Z given $Z < X$ is stochastically increasing in β , and (ii) the conditional distribution of $X - Z$ given $Z = z, Z < X$ is stochastically decreasing in β .

Consider next $Y = \min(X, Z)$, which is the actual operation time of the component. The following results are therefore of practical interest, and may in addition shed light on the influence of the parameters of the repair alert model. We therefore state them formally as a theorem.

Theorem 5 *Suppose (X, Z) has a repair alert distribution with parameters q , F_X and G . Then,*

$$P(Y \leq t) = F_X(t) + qG(t) \int_t^\infty \frac{f_X(y)}{G(y)} dy \quad (10)$$

$$E(Y) = E(X) - qE \left[\frac{M(X)}{G(X)} \right], \text{ where } M(x) = \int_0^x G(t) dt. \quad (11)$$

Furthermore, if $G(t) = t^\beta$, then

$$E(Y) = E(X) \left(1 - \frac{q}{\beta + 1} \right) \quad (12)$$

Proof: Equation (10) follows from Theorem 2 and the simple fact that $P(Y \leq t) = F_X^*(t) + F_Z^*(t)$. By integration we next obtain (11) from (10), while (12) is a special case of (11).

We close the section by a simple illustration of how the parameters q and β (assuming $G(t) = t^\beta$ for simplicity) influence the long run cost per time unit under the repair alert model.

Let C_{PM}, C_F be costs of PM and failure, respectively, for a single sojourn. Assume now that following an event (PM or failure), the operation is restarted with a component assumed to be as good as new, and that this process continues. This leads to a sequence of observations of (Y, δ) , which we shall assume are independent and identically distributed. The theory of renewal reward processes (e.g. Ross, 1983, p. 78) implies that the expected cost per unit time in the long run equals the expected cost per sojourn divided by the expected length of a sojourn, i.e.

$$\frac{qC_{PM} + (1 - q)C_F}{E(X) \left(1 - \frac{q}{\beta + 1} \right)}$$

where we used (12).

This is a decreasing function of β , which is reasonable in the light of Theorem 4. On the other hand, it is a decreasing function of q provided $\beta > C_{PM}/(C_F - C_{PM})$. This last inequality is likely to hold in many practical cases since the right hand side will usually be much less than 1, while β should for a competent maintenance crew be larger than 1. Thus a high value of q is usually preferable.

6 Statistical inference in the repair alert model

Let $(y_1, \delta_1), (y_2, \delta_2), \dots, (y_N, \delta_N)$ be N i.i.d. observations of $(Y, \delta) \equiv (\min(X, Z), I(Z < X))$.

In practice some observations may be censored by external causes, so that it is only known that $Y > c$ for some censoring time c and not known whether there would be a PM or failure. For simplicity of exposition we will not assume this to be the case here (see e.g. Crowder, 2001, Ch. 2) for treatment of censorings). Following Bedford and Cooke (2001, Section 9.5) we let the observations be given on the form $x_1, \dots, x_m$ and $z_1, \dots, z_n$, which are, respectively, the observed times to failure and the observed times for PM.

For practical illustration we use data from Mendenhall and Hader (1958). The data are times to failure for ARC-1 VHF communication transmitter-receivers of a single commercial airline. They will later be referred to as the VHF-data. Units which failed were removed from the aircraft for maintenance. But, in some cases the apparent failures were unconfirmed, exhibiting satisfactory operation upon arrival at the maintenance centre. Thus, the failure times can be divided in two groups, unconfirmed, Z , and confirmed failures, X . There are $m = 218$ observations of X and $n = 107$ observations of Z . It is noted in Mendenhall and Hader (1958) that the data are in fact externally censored at time 630. However, for the analysis here we shall disregard this information and treat the set as a complete set of $N = 325$ cases, in each of which either X or Z is observed.

Suppose we want to fit a repair alert model to the data. By Theorem 3 we need to have $\tilde{F}_X(t) < \tilde{F}_Z(t)$ for all $t > 0$. As noted by Cooke (1993), this is equivalent to $\Phi(0) > \Phi(t)$ for all t , where $\Phi(t) = P(Z < X | X > t, Z > t)$ is the probability of PM conditional on no event by time t .

Figure 2 shows both the empirical versions of the conditional survival functions $1 - \tilde{F}_X(t)$ and $1 - \tilde{F}_Z(t)$, together with the empirical version of $\Phi(t) = P(Z < X | X > t, Z > t)$. The required inequalities (see previous paragraph) are seen to hold for the empirical functions, except possibly for some observations below time 100 where the two survival curves are very close. However, we conclude that it is indeed meaningful to fit a repair alert model. Formal tests for investigations of this kind are considered by Dewan et al. (2002).

6.1 Nonparametric estimation

In this subsection we suggest simple nonparametric estimators of q , F_X and G for the repair alert model.

Fig. 2. Estimated conditional survival functions for X , thin line, Z , thick line, estimated $\Phi(t) = P(Z < X|X > t, Z > t)$ ($- \cdot -$), and the estimated conditional survival function for a model where X, Z are independent and exponentially distributed (see Stove, 2002).

Since $q = P(Z < X)$, its natural estimator is $\hat{q} = n/N$. For the VHF-data this equals $\hat{q} = 107/325 = 0.3292$.

From (1) we know that F_X under the repair alert model equals the conditional distribution function $\tilde{F}_X$. The natural estimator $\hat{F}_X$ of F_X is thus the empirical distribution function based solely on the $x_1, \dots, x_m$, i.e.

$$\hat{F}_X(t) = \frac{i}{m} \text{ for } x_i \leq t < x_{i+1}, \quad i = 0, 1, \dots, m \quad (13)$$

where we put $x_0 = 0$, $x_{m+1} = \infty$.

Similarly we may estimate $\tilde{F}_Z$ by the empirical distribution function based on $z_1, \dots, z_n$, namely

$$\hat{\tilde{F}}_Z(t) = \frac{j}{n} \text{ for } z_j \leq t < z_{j+1}, \quad j = 0, 1, \dots, n,$$

where $z_0 = 0$, $z_{n+1} = \infty$. We will use this estimator to obtain an estimator

of the cumulative repair alert function G . Note first that by substituting $y = \tilde{F}_Z(w)$ in (6), we can write

$$G(t) = \exp\left\{ \int_{\tilde{F}_Z(t_0)}^{\tilde{F}_Z(t)} \frac{dy}{y - F_X(\tilde{F}_Z^{-1}(y))} \right\}. \quad (14)$$

The natural estimator of this function is obtained by replacing F_X and $\tilde{F}_Z$ by the corresponding estimated functions. Here is a slight modification of this direct procedure. The clue of the method is to substitute only $\hat{F}_X$ first, and then do the integration before finally substituting $\hat{\tilde{F}}_Z$ for $\tilde{F}_Z$. The details are as follows:

From (13) we get

$$\hat{F}_X(\tilde{F}_Z^{-1}(y)) = \frac{i}{m} \text{ for } \tilde{F}_Z(x_i) \leq y < \tilde{F}_Z(x_{i+1}), \quad i = 0, 1, \dots, m.$$

and hence the exponent of equation (14), with $t_0 = x_1$ and $t = x_j$, is

$$\int_{\tilde{F}_Z(x_1)}^{\tilde{F}_Z(x_j)} \frac{dy}{y - \hat{F}_X(\tilde{F}_Z^{-1}(y))} = \sum_{i=1}^{j-1} \int_{\tilde{F}_Z(x_i)}^{\tilde{F}_Z(x_{i+1})} \frac{dy}{y - i/m} = \sum_{i=1}^{j-1} \log \frac{\tilde{F}_Z(x_{i+1}) - i/m}{\tilde{F}_Z(x_i) - i/m} \quad (15)$$

At this stage we tacitly assume that $\tilde{F}_Z(x_i) > i/m$ for all i .

Inserting (15) in equation (14), and replacing $\tilde{F}_Z$ by $\hat{\tilde{F}}_Z$, we obtain the estimator

$$\hat{G}(x_j) = \prod_{i=1}^{j-1} \frac{\hat{\tilde{F}}_Z(x_{i+1}) - i/m}{\hat{\tilde{F}}_Z(x_i) - i/m} = \prod_{i=1}^{j-1} \frac{\#\{z_k : z_k \leq x_{i+1}\}/n - i/m}{\#\{z_k : z_k \leq x_i\}/n - i/m} \quad (16)$$

defined at the points x_j (where by construction we have $\hat{G}(x_1) = 1$).

In practice it may happen that for some i , $\#\{z_k : z_k \leq x_i\}/n \leq i/m$. In this case we suggest to put the corresponding factor of (16) equal to 1.

Figure 3 shows a graph of the described estimator for the VHF-data, with $\log \hat{G}(x_j)$ plotted against $\log x_j$. The motivation for the log is to check whether the parametrization $G(t) = t^\beta$ is plausible. In that case we will have $\log G(t) = \beta \log t$, so we would expect plots of $\log \hat{G}(x_j)$ against $\log x_j$ to be approximately a straight line with slope β . This is approximately so in Figure 3.

Fig. 3. The estimated $\log \hat{G}(x_j)$ plotted against $\log(x_j)$, Mendenhall and Hader (1958) data

Based on the figure, we would roughly estimate the slope of the curve to be around 5, which is hence our first guess of β .

In fact we may also obtain a quick estimate of β from equation (9). We then estimate $E(X)$ by the mean of the x_i and $E(Z|Z < X)$ by the mean of the z_j . Solving (9) for β gives the estimate 4.98 for the VHF-data, which is a reasonable estimate in view of the plot.

6.2 Parametric estimation

In this section we assume the special parametric model where X is exponentially distributed with $f_X(x) = \lambda e^{-\lambda x}$, while $G(t) = t^\beta$.

Following Crowder (2001, Chapter 2), the contributions to the likelihood from an observation is given by the subdensity function at the observed time. Thus, using Theorem 2, the likelihood contribution from an x_i is

$$f_X^*(x_i) = (1 - q)f_X(x_i) = (1 - q)\lambda e^{-\lambda x_i}$$

while the contribution from z_j is

$$f_Z^*(z_j) = qg(z_i) \int_{z_i}^{\infty} (f_X(t)/G(t)) dt = q\lambda\beta(\lambda z_i)^{\beta-1} \int_{\lambda z_i}^{\infty} w^{-\beta} e^{-w} dw$$

The total likelihood for the data $x_1, \dots, x_m$ and $z_1, \dots, z_n$ is obtained as the product for each data point. Taking the logarithm we obtain the log-likelihood function

$$\begin{aligned} l(\lambda, \beta, q) = & m \log(1 - q) + n \log q + (n + m) \log \lambda + n \log \beta - \lambda \sum_{i=1}^m x_i \\ & + \sum_{i=1}^n (\beta - 1) \log(\lambda z_i) + \sum_{i=1}^n \log \left(\int_{\lambda z_i}^{\infty} w^{-\beta} e^{-w} dw \right) \end{aligned} \quad (17)$$

Note that if there were externally censored observations, then they would each contribute to the log-likelihood by adding $\log P(Y > c)$, which can be computed from (10). Here c is the censoring time.

The maximum likelihood estimators can now be found by maximizing the log-likelihood (17). This has to be done numerically except for the parameter q which is readily estimated by $\hat{q} = n/(m + n)$ as could be expected.

For computation of the maximum likelihood estimators for β and λ it turns out that the EM-algorithm (Dempster, Laird and Rubin, 1977) is useful. The main idea is to augment the data artificially in order to obtain a more tractable likelihood function, for which there may for example exist explicit expressions for the maximum likelihood estimators. This is the so called M-step (maximization step). The M-step alternates in an iterative manner with the E-step, in which we compute the conditional expectation of the augmented likelihood function conditional on the data.

Here we shall assume that we observe X also when $Z < X$. It is practical to change slightly the meaning of the x_i and z_j . We now assume that there are N triples (x_i, z_i, δ_i) . Here $\delta_i = 0$ if $x_i < z_i$, in which case we observe only x_i, δ_i , and $\delta_i = 1$ if $z_i < x_i$, in which case we observe the whole triple (x_i, z_i, δ_i) . This augments our data, and the augmented likelihood becomes

$$L_A(\lambda, \beta, q) = \prod_{i=1}^N \left\{ \lambda e^{-\lambda x_i} (1 - q)^{1-\delta_i} q^{\delta_i} \left(\frac{\beta z_i^{\beta-1}}{x_i^\beta} \right)^{\delta_i} \right\}$$

which by taking the logarithm gives the augmented log-likelihood,

$$\begin{aligned}
l_A(\lambda, \beta, q) = & N \log \lambda - \lambda \sum_{i=1}^N x_i + N \log(1 - q) - \log(1 - q) \sum_{i=1}^N \delta_i + \log q \sum_{i=1}^N \delta_i \\
& + \log \beta \sum_{i=1}^N \delta_i + (\beta - 1) \sum_{i=1}^N \delta_i \log z_i - \beta \sum_{i=1}^N \delta_i \log x_i.
\end{aligned} \tag{18}$$

We consider the M-step first, where we find the maximum likelihood estimators from (18). Then we consider the E-step where we replace the unobserved data with their expected values, conditional on the observed data and the current parameter estimates.

- M-step: This gives us explicit expressions for the maximum likelihood estimators:

$$\hat{q} = \frac{\sum_{i=1}^N \delta_i}{N} \tag{19}$$

$$\hat{\lambda} = \frac{N}{\sum_{i=1}^N x_i} \tag{20}$$

$$\hat{\beta} = \frac{\sum_{i=1}^N \delta_i}{\sum_{i=1}^N \delta_i \log(x_i/z_i)} \tag{21}$$

- E-step: By inspection of (18) we conclude that we need to compute expected values of x_i and $\log x_i$ for the i for which $z_i < x_i$. For this we need the conditional density of X given $Z < X, Z = z$. The basic properties of the repair alert model imply that this density is proportional to $f_X(x)/G(x)$ (see discussion before Theorem 4). For the special model considered here this gives

$$f(x|Z < X, Z = z) = \frac{x^{-\beta} e^{-\lambda x}}{\lambda^{\beta-1} \int_{\lambda z}^{\infty} w^{-\beta} e^{-w} dw}$$

By computing the required expectations and inserting them into (19-21) we get the following iterative expressions:

$$\begin{aligned}
\hat{q} &= \frac{n}{N} \\
\hat{\lambda}_{j+1} &= \frac{N}{\sum_{i=1}^m x_i + (1/\hat{\lambda}_j) \sum_{i=m+1}^N \frac{\int_{\hat{\lambda}_j z_i}^{\infty} w^{-(\hat{\beta}_j-1)} e^{-w} dw}{\int_{\hat{\lambda}_j z_i}^{\infty} w^{-\hat{\beta}_j} e^{-w} dw}} \\
\hat{\beta}_{j+1} &= \frac{n}{\sum_{i=m+1}^N \left[\frac{\int_{\hat{\lambda}_j z_i}^{\infty} \log(w) w^{-\hat{\beta}_j} e^{-w} dw}{\int_{\hat{\lambda}_j z_i}^{\infty} w^{-\hat{\beta}_j} e^{-w} dw} - \log(\hat{\lambda}_j z_i) \right]}
\end{aligned}$$

The resulting estimates for the VHF-data are given in Table 1, where we

Table 1

Maximum likelihood estimates and approximate 95% confidence intervals for parametric repair alert model for VHF-data.

Parameter	Estimate	Lower bound	Upper bound
q	0.3292	0.2781	0.3803
λ	$4.458 \cdot 10^{-3}$	$3.916 \cdot 10^{-3}$	$5.069 \cdot 10^{-3}$
β	8.9809	3.0345	∞

also include bounds for approximate 95% confidence intervals (Støve, 2002). It is remarkable that the interval for β extends all the way to infinity. In fact also the maximum likelihood estimate is high, 8.98, compared to the moment type estimator of the previous subsection which gave the value 4.98. For an interpretation of these results on β in terms of repair alert, it may help to consider Figure 1 and the corresponding discussion in Section 5. Indeed, assuming that the requirements of random signs hold true, we conclude that the “PM” for these data censor the “failures” in a near optimal way as regards the repair alert function.

7 Concluding remarks

We have proposed and studied the so called repair alert model which describes the dependence between time for PM (Z) and time to failure (X) for a component. The model is a special case of random signs censoring, with the aim to model in more detail the behaviour of the maintenance crew with respect to failures of the component. This is done via the (cumulative) repair alert function G , which turns out to be uniquely given from the observed subdistribution functions. On the other hand, the random signs property guarantees the identifiability of the marginal distribution of X .

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References

- [1] Bedford, T., Cooke, R., 2001. Probabilistic risk analysis: Foundations and methods. Cambridge University Press, Cambridge.
- [2] Cooke, R.M., 1993. The total time on test statistics and age-dependent censoring. *Statistics and Probability Letters* 18, 307-312.
- [3] Cooke, R.M., 1996. The design of reliability databases, Part I and II. *Reliability Engineering and System Safety* 51, 137-146 and 209-223.
- [4] Crowder, M., 2001. Classical Competing Risks. Chapman and Hall/CRC, Boca Raton.
- [5] Dempster, A. P., Laird, N. M. Rubin, D. B., 1977. Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society, Ser. B* 39, 1-38.
- [6] Dewan, I., Deshpande, J.V., Kulathinal, S.B., 2002. On testing dependence between time to failure and cause of failure via conditional probabilities. Technical Report isid/ms/2002/17, Statistics & Mathematics Unit, Indian Statistical Institute, Delhi Centre.
- [7] Langseth, H., Lindqvist, B.H., 2003. A maintenance model for components exposed to several failure mechanisms and imperfect repair. In: Lindqvist, B.H., Doksum, K.A. (Eds.), *Mathematical and Statistical Methods in Reliability. Series on Quality, Reliability and Engineering Statistics, Vol. 7*, World Scientific Publishing, Singapore, 415-430.
- [8] Langseth, H., Lindqvist, B.H., 2004. Competing risks for repairable systems: A data study. Submitted to *Journal of Statistical Planning and Inference*, Special Issue on Competing Risks.
- [9] Mendenhall, W., Hader, R.J., 1958. Estimation of parameters of mixed exponentially distributed failure time distributions from censored life test data. *Biometrika* 45, 504-520.
- [10] Ross, S.M., 1983. *Stochastic Processes*. Wiley, New York.
- [11] Støve, B., 2002. Competing risk models. Master Thesis, Department of Mathematical Sciences, Norwegian University of Science and Technology.
- [12] Tsiatis, A., 1975. A nonidentifiability aspect of the problem of competing risks. *Proceedings of the National Academy of Sciences, USA*, 72, 20-22.