THE BRACHISTOCHRONE – A PARTIAL SOLUTION
(Trotman p. 66 – 68)

Kinetic energy =
lost potential energy:

\[ mgx = \frac{1}{2}mv^2(x) \]

Path:

\[ ds = \sqrt{1 + (y'(x))^2} \, dx \]

Travel time:

\[ T(y) = \int_{0}^{x_i} \frac{ds}{v} = \int_{0}^{x_i} \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{2gx}} \, dx \]

Problem: The opt. path is not necessarily monotone in \( x \).
$$f(x, y') = \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gx}}$$

$$\sqrt{1 + y'(x)^2} \text{ is strongly convex}$$

$$\sqrt{2gx} > 0$$

$$T(y) \text{ is strictly convex}$$

Euler equation: \[
\frac{d}{dx} \left[ f_y'(x, y') \right] = 0
\]

$$\frac{y'(x)}{\sqrt{2gx} \sqrt{1 + y'(x)^2}} = \frac{1}{\tilde{c}} \quad \Rightarrow \quad \frac{y'(x)}{\sqrt{1 + y'(x)^2}} = \frac{\sqrt{x}}{c}$$
Equation is only meaningful for \( y'(x) > 0 \)

and then \( c > \sqrt{x} \)

Square equation: \[
\frac{y'(x)^2}{1 + y'(x)^2} = \frac{x}{c^2}
\]

or \[
y'(x)^2 = \frac{x}{c^2 - x}
\]

Implicit solution: \[
y(x) = \int_0^x \sqrt{\frac{t}{c^2 - t}} \, dt
\]

Set \( t = \frac{c^2}{2} (1 - \cos \theta) \)
By a small trick using trig. substitutions,

\[
x(\theta) = c^2 (1 - \cos \theta) \\
y(\theta) = c^2 (\theta - \sin \theta)
\]

\(c := \sqrt{2c}\)
There is always a (unique) cycloid of the form
\[ x(\theta) = c^2 (1 - \cos \theta) \]
\[ y(\theta) = c^2 (\theta - \sin \theta) \]
for all \((x_1, y_1), x_1, y_1 > 0\):

However, it is possible to write \( y = y(x) \) only for \( \theta \leq \pi \)

(But that solves our problem in that case, since then \( T(y) \) is strictly convex)
The solution is a cycloid in general!

(Proved later in Troutman)