• By using variational calculus it is possible to formulate and analyze a simple mathematical model of how one should prepare for an exam.

• Since the optimal solution appears to reproduce the typical student preparation, it is not necessary optimal for long term knowledge!
1 The Model

- Learning depends on the effort, $I^*(t^*)$.

- Effort below a certain level $I_0$ gives no significant contribution to the insight/knowledge!

- Very high effort is not necessary very efficient!

Acquired knowledge per time unit as a function of the effort:

$$D(I^*) = \kappa \ln^+ \left( \frac{I^*}{I_0} \right)$$

$$\ln^+ (x) = \begin{cases} 
0, & x \leq 1, \\
\ln x, & x > 1.
\end{cases}$$

The learning ability coefficient, $\kappa$
Figure 1: Acquired knowledge per time unit as a function of the effort.

Acquired knowledge at time $t^*$: $L^*(t^*)$

The breakdown of knowledge:

$$\frac{dL^*(t^*)}{dt^*} = -\frac{L^*(t^*)}{\alpha},$$

Breakdown time constant: $\alpha$. 
Dynamic model for the knowledge:

\[
\frac{dL^*(t^*)}{dt^*} = \kappa \ln^+ \left( \frac{I^*}{I_0} \right) - \frac{L^*(t^*)}{\alpha}
\]

- No use in an effort \( I^* \) less than \( I_0 ! \)

- The students start without any knowledge about the subject: \( L^*(0) = 0 \).

At the time of the exam (at time \( t^* = T \)), the target is \( L^*(T) = p \), that is, the curriculum.

**AIM:** How do we reach the target with the minimal total effort,

\[
J^*(I^*) = \int_0^T I^*(t^*)dt^*.
\]
2 Scaling and Problem Formulation

A scale is a natural measuring stick for a variable (more about this in the Mathematical modelling course).

Reasonable (but not the only) scales:

\[ L^* = pL, \]
\[ I^* = I_0I, \]
\[ t^* = \alpha t, \]
\[ T = \alpha a. \]

The differential equation in dimensionless form:

\[ \frac{dL(t)}{dt} = \frac{1}{\mu} \ln^+ I(t) - L(t), \]

Dimensionless curriculum: \( \mu = \frac{p}{\alpha K} \)

Dimensionless reading period: \( \alpha \)
The expression for total effort:

\[ J^*(I^*) = \alpha I_0 \int_0^\alpha I(t) \, dt, \]

We therefore consider

\[ J(I) = \int_0^\alpha I(t) \, dt. \]

**PROBLEM:**

\[ \min_I J(I) = \min_I \int_0^\alpha I(t) \, dt, \]

when

\[ \frac{dL(t)}{dt} = \frac{1}{\mu} \ln^+ I(t) - L(t), \]

\[ L(0) = 0, \]

\[ L(\alpha) = 1. \]

How do we solve it??
The general solution of the differential equation:

\[ L(t) = Ce^{-t} + \frac{1}{\mu} \int_{\tau=0}^{t} \ln^+ (I(\tau)) e^{-(t-\tau)} d\tau. \]

Since \( L(0) = 0 \), \( C = 0 \).

**FINAL FORMULATION:**

\[ \min \int_{0}^{a} I(t) dt, \]

when

\[ G(I) = \int_{0}^{a} \ln^+ (I(t)) e^{t-a} dt = \mu. \]
3 Solution and Analysis

- Both functionals have the standard form

- The functional $J$ is convex

- Since $d^2 \ln^+(x)/dx^2 < 0$ for $x > 1$, and $e^{t-a} > 0$, then the kernel $\ln^+(I(t)) e^{t-a}$ is strongly concave. Thus, $-G(I)$ is strictly convex when $I > 1$.

- The Lagrange functional

\[
\mathcal{L}(I) = J(I) - \lambda G(I),
\]

is strictly convex for $\lambda > 0$, as long as $I(t) \geq 1$

- Any solution of $\delta \mathcal{L}(I,v) = 0$ is then a global minimum (Troutman, Theorem 3.16).

**COMPLICATION:** If $\mu$ is too small, it does not pay to read the whole period $\alpha$!
3.1 Case A: $I(t) > 1$ for all $t \in [0, a]$.

$$
\mathcal{L}(I) = \int_0^a \left( I(t) - \lambda \ln[I(t)] e^{t-a} \right) dt.
$$

The Euler equation:

$$
\frac{\partial}{\partial I} \left[ I - \lambda (\ln I) e^{t-a} \right] = 1 - \frac{\lambda}{I} e^{t-a} = 0.
$$

Solution:

$$
I(t) = \lambda e^{t-a}.
$$

The constant $\lambda$ is found from

$$
\mu = \int_0^a (\ln \lambda + (t - a)) e^{t-a} dt
= \ln \lambda \cdot (1 - e^{-a}) + (a + 1)e^{-a} - 1,
$$

or

$$
\ln \lambda = \frac{\mu e^a + e^a - a - 1}{e^a - 1}.
$$

This is an optimal solution if

$$
I(t) = \lambda e^{t-a} \geq 1 \text{ for all } t \in [0, a],
$$
that is

\[ \lambda e^{-a} \geq 1, \]

or

\[ \ln \lambda \geq a. \]

This holds when

\[ \mu \geq a - 1 + e^a. \]

Let \( a_0 \) be the solution of

\[ \mu = a_0 - 1 + e^{a_0} \]

\((a_0 \text{ is the limiting length of the reading period for this case})\)

If \( a \leq a_0 \), then the optimal effort function, assuming that we choose to read the whole period \( a \), is

\[ I_{opt}(t) = \frac{\lambda}{e^{a} e^{t}}, \quad \lambda = \exp \left( \frac{\mu e^a + e^a - a - 1}{e^a - 1} \right) \]

An exponential increase towards the exam!
The differential equation is

$$\frac{dL(t)}{dt} + L(t) = \frac{1}{\mu} \ln \left( \frac{\lambda e^t}{e^a} \right) = \frac{\ln \lambda - a + t}{\mu}$$

with solution

$$L(t) = \frac{\ln \lambda - a - 1 + t}{\mu} - \frac{\ln \lambda - a - 1}{\mu} e^{-t}$$

(Yes! $L(0) = 0$ and $L(a) = 1$!)

3.2 Case B: $a > a_0$

- $\mu < a - 1 + e^a$

- $I(t) = \lambda e^{t-a} < 1$ in part of the interval

- The Lagrangian, $\mathcal{L}(I) = J(I) - \lambda G(I)$ is no longer convex (cf. Fig. 1)

**Smart idea:** *Do not start the reading until the time to the exam is $a_0$. Then follow the optimal solution above.*

Thus, do not start too early because you then forget too much during the reading period!

**Smarter idea:** *What is the optimal length of the reading period $a$ among all with $a \leq a_0$? (For each such $a$, we use the optimal strategy)*
Effort as a function of $a$:

$$J(a) = \int_0^a I(t)dt$$

$$= \int_0^a \lambda e^{t-a}dt$$

$$= \lambda (1 - e^{-a}).$$

Thus,

$$\ln J = \ln \lambda + \ln(1 - e^{-a})$$

$$= \frac{\mu e^a + e^a - a - 1}{e^a - 1} + \ln(1 - e^{-a})$$

$$= 1 + \frac{\mu e^a - a}{e^a - 1} + \ln(1 - e^{-a})$$

$$\frac{d \ln J}{da} = e^a \frac{a - \mu}{(e^a - 1)^2},$$

which has a minimum for $a = \mu$.

For the optimal situation $a = \mu$:

$$I(t) = e^{t+1},$$

$$L(t) = \frac{t}{a}. \quad (1)$$
Figure 2: Total effort in order to learn the curriculum as a function of the length of the reading period ($\mu = 0.5a_0$).
Figure 3: Some other optimal cases (\(Tid = \text{Time}, \ Innsats = \text{Effort}, \ Stoffmengde = \text{Curriculum}\)).
4 Summary

Parameters:

Length of the semester: $T$

Low effort limit: $I_0$

The learning ability coefficient: $\kappa$

Forgetfulness time constant: $\alpha$

The curriculum: $p$

Dimensionless parameters:

Dimensionless curriculum: $\mu = \frac{p}{\kappa \alpha}$

The dimensionless limiting period $a_0$ (solution of $\mu = a - 1 + e^a$)
The optimal starting time given by $a_{opt} = \mu$

Overall optimal starting time ($\mu = a_{opt}$) given by $\frac{p}{\kappa \alpha} = a_{opt}$, or

$$T_{opt} = \frac{p}{\kappa}.$$ 

Define also

$$T_0 = \alpha a_0$$
5 Conclusions

1. Read continuously!

2. If $T_{\text{opt}} \leq T$, do not start the reading before $T_{\text{opt}}$ and then follow the optimal strategy, $I(t) = e^{t+1}$, that is,

$$I_{\text{opt}}(t^*) = I_0 \exp \left( \frac{t^*}{\alpha} + 1 \right).$$

3. If $T_{\text{opt}} \geq T$, start at once and follow the strategy

$$I_{\text{opt}}(t^*) = I_0 \lambda \exp \left( \frac{t^* - T}{\alpha} \right),$$

$$\lambda = \exp \left( \frac{\mu e^a + e^a - a - 1}{e^a - 1} \right).$$

4. Never start before $T_0$!
5. It is reasonable to start so that

\[ T_{opt} \leq T_{start} \leq \min(T, T_0) \]

6. Do not wait until \( T_{start} < T_{opt} \)!

7. For the teacher: By keeping the curriculum large enough, the students will find that they have to study the whole term with an exponential increase towards the exam!