1 Optimization of production (Troutman, p. 72 - 74)

- $L(t)$: Inventory ($=$ amount in storage)

- $P(t)$: Production per time unit.

- $S(t)$: Sales per time unit.

System model:

$$\frac{dL(t)}{dt} = P(t) - S(t) - \alpha L(t)$$

($-\alpha L$: loss at the storage)

- $\mathcal{S}(t)$: Sales prognosis

The *optimal* inventory ($\mathcal{L}$) and production ($\mathcal{P}$) are the ones matching the sales prognosis:

$$\frac{d\mathcal{L}(t)}{dt} = \mathcal{P}(t) - \mathcal{S}(t) - \alpha\mathcal{L}(t)$$

At $t = 0$, the inventory is off the ideal inventory, $L(0) = L_0 \neq \mathcal{L}(0)$.

How do we control the production $P(t)$ so as to minimize the extra cost of being off the ideal?

1.1 Cost functional

(what we suffer from not being on the ideal track)

- $L(t) - \mathcal{L}(t)$: Deviation from optimal inventory
- $P(t) - \mathcal{P}(t)$: Deviation from opt. production rate

$$C(P) = \int_0^T \left[\beta^2 (L(t) - \mathcal{L}(t))^2 + (P(t) - \mathcal{P}(t))^2\right] dt$$

This is an *Optimal Control* problem:

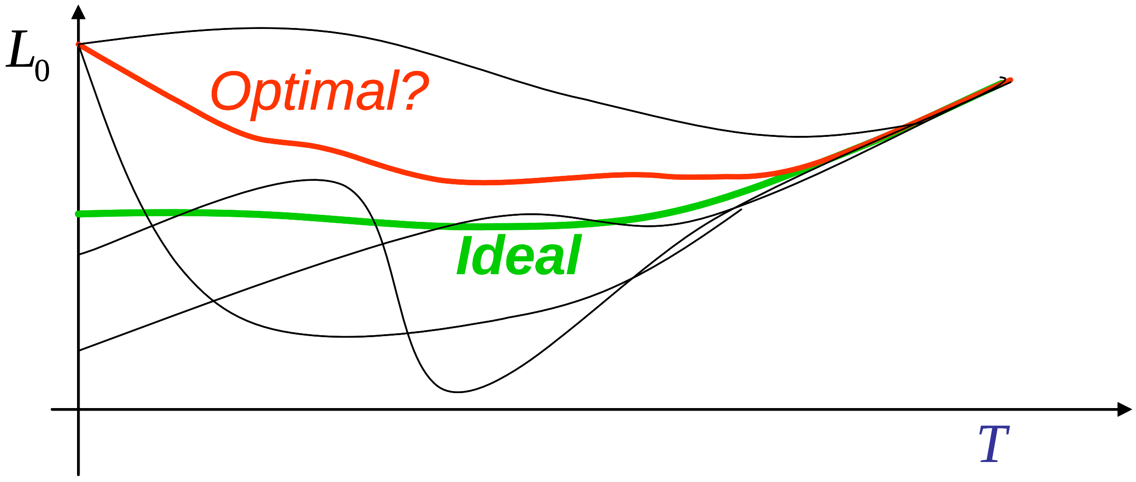
$$\min_P C$$
$$L'(t) = P(t) - \mathcal{S}(t) - \alpha L(t),$$
$$L(0) = L_0.$$

- The variable $P(t)$ is our *control variable*

- We expect the sales prognosis $\mathcal{S}(t)$ to be true, and try to adjust the inventory towards the optimal $\mathcal{L}(t)$

- We are forced to follow the system dynamics

$$\frac{dL(t)}{dt} = P(t) - \mathcal{S}(t) - \alpha L(t)$$



We are starting off the ideal curve (which we know), and want to minimize the cost of getting there.

Observe that P is dependent on L, and if we determine L, P may be found as

$$P = L' + \alpha L + \mathcal{S}$$

Let $y = L(t) - \mathcal{L}(t)$. Then, since

$$L' = P - \mathcal{S} - \alpha L,$$
$$\mathcal{L}' = \mathcal{P} - \mathcal{S} - \alpha \mathcal{L},$$

we have

$$L' - \mathcal{L}' = P - \mathcal{P} - \alpha \left(L' - \mathcal{L}' \right),$$

or

$$P - \mathcal{P} = \frac{dy}{dt} + \alpha y.$$

This is inserted into the cost functional:

$$C(y) = \int_0^T \left[\beta^2 y^2 + \left(y' + \alpha y\right)^2\right] dt$$

Note that

$$\begin{aligned} C(y) &= \int_0^T \left[\beta^2 y^2 + \left(y' + \alpha y\right)^2\right] dt \\ &= \int_0^T \left[\left(\beta^2 + \alpha^2\right) y^2 + y'^2\right] dt + \int_0^T 2\alpha y y' dy \\ &= \int_0^T \left[\left(\beta^2 + \alpha^2\right) y^2 + y'^2\right] dt + \alpha y^2(T) - \alpha y^2(0) \end{aligned}$$

The first term is strictly convex, the second is convex, and the last is just a constant ($= -\alpha \left(L(0) - \mathcal{L}(0)\right)^2$). Thus, C is strictly convex.

1.2 Solution

The solution is found by solving the Euler equation for C,

$$\frac{d}{dx}f_{y'} - f_y = \frac{d}{dx}\left[2\left(y' + \alpha y\right)\right] - \left[2\beta^2 + 2\left(y' + \alpha y\right)\alpha\right]$$
$$= y'' - \left(\alpha^2 + \beta^2\right)y = 0.$$

with a fixed boundary at $t = 0$,

$$y(0) = y_0 = L_0 - \mathcal{L}_0.$$

and a natural condition at $t = T$ (we have no particular restriction on y at T):

$$f_{y'}(y(T)) = 2\left(y'(T) + \alpha y(T)\right) = 0.$$

With $\gamma^2 = \left(\alpha^2 + \beta^2\right)$, the problem may be formulated as

$$y'' - \gamma^2 y = 0,$$
$$y(0) = y_0,$$
$$y'(T) + \alpha y(T) = 0.$$

From the general solution of the Euler eqn.,

$$y(t) = A \exp(\gamma t) + B \exp(-\gamma t),$$

it is easy to find the unique optimal solution

$$y^*(t)) = y_0 \frac{e^{\gamma t} + \rho e^{-\gamma t}}{1+\rho},$$
$$\rho = \frac{\gamma + \alpha}{\gamma - \alpha} e^{2\gamma T}.$$

The minimal value for the functional may also be computed:

$$C(y^*) = y_0^2 \left[\frac{\gamma(\rho - 1)}{\rho + 1} - \alpha \right].$$

1.3 Example: The Ice Cream Factory

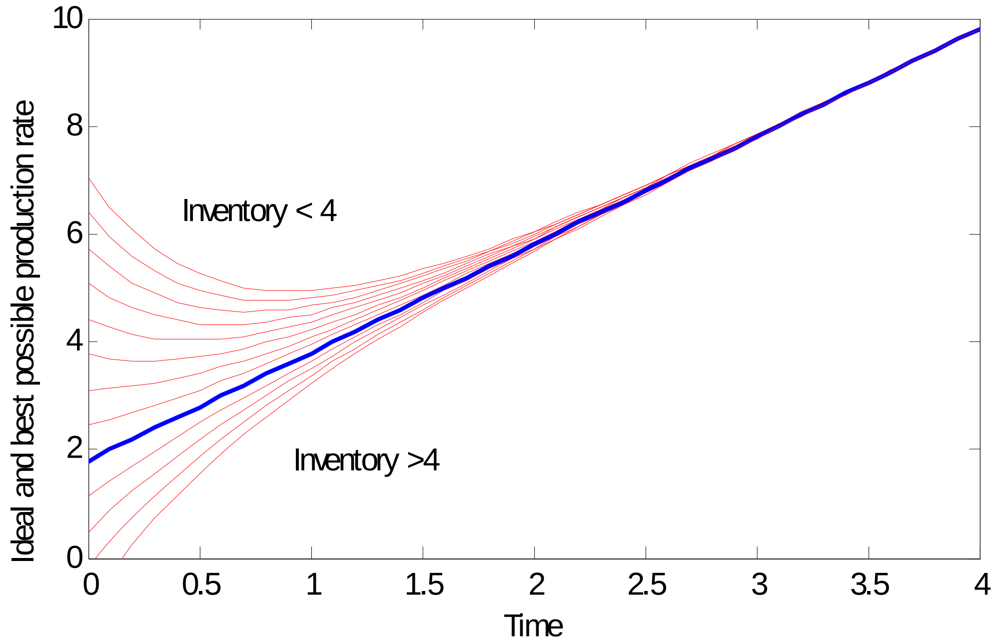
The factory is planning for the summer. $t = 0$ is April 1st, $t = 4$ is July 31st.

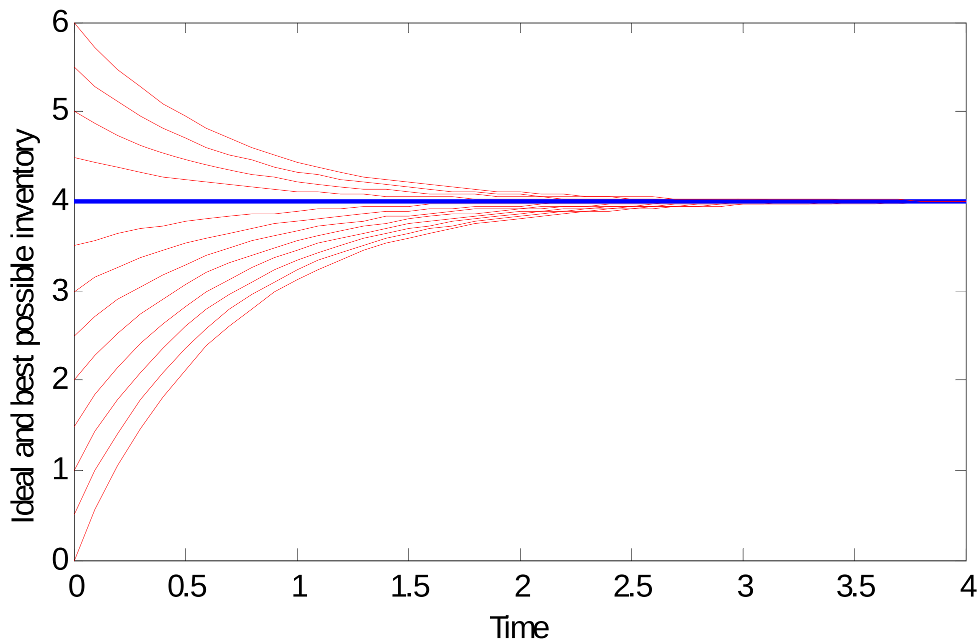
- Sales prediction: $\mathcal{S}(t) = 1 + t$.

- Optimal inventory: $L(t) = 4$

- The optimal production rate for $L = 4$ (and $L'(t) = 0$): $\mathcal{P}(t) = \mathcal{S}(t) + \alpha \times 4$.

Let the decay rate, $\alpha = 0.1$, and the constant in the cost functional, $\beta = 1.5$.


Ideal and best possible production rate
Time
Inventory < 4
Inventory >4
0
0.5
1
1.5
2
2.5
3
3.5
4
0
2
4
6
8
10


Ideal and best possible inventory
Time
0
0.5
1
1.5
2
2.5
3
3.5
4
0
1
2
3
4
5
6