

Alternative Integration Methods for Problems in Structural Dynamics

B. Owren* H. H. Simonsen*

Abstract

Runge-Kutta methods for the time integration of the equations of motion in structural dynamics are presented. The methods belong to a class called SDIRK (singly-diagonally-implicit Runge-Kutta) methods. The computational cost needed per step is comparable to Newmark methods with α damping. The methods proposed are L-stable which means that they instantly damp out the higher modes in the solution. The numerical dissipation can be controlled by a parameter. Both second and third order methods are presented. Some characteristics of the methods are compared with the Newmark method with α damping.

1 Introduction

Initial value problems of the form

$$y'' = g(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \quad (1)$$

appear in many applications, for instance in structural dynamics. Here y, y' and y'' are all vectors in R^m while $g : R \times R^m \times R^m \rightarrow R^m$ defines m (nonlinear) equations.

Sometimes, the definition of (1) is fully implicit, in which case it is interesting to consider a linearized version,

$$My'' + Cy' + Ky = F(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \quad (2)$$

where M, C and K are $m \times m$ matrices and M is non-singular. This equation can be thought of as a linearized version of the implicit system

$$F(t, y, y', y'') = 0. \quad (3)$$

However, another approach is to discretize (3) directly and thereby obtain a nonlinear system of equations which must be solved for each time step. For instance, if this nonlinear system is solved by using Newton iterations, the matrices M, C and K will appear as partial derivatives of the function F .

In a paper by Hilber et al. [3], the authors consider (2) with $C = 0$. For the applications they have in mind, they require methods that possess numerical dissipation for

*SINTEF, Industrial Mathematics, N-7034 Trondheim, Norway

high frequency components in the solution. It is also important that the methods yield high accuracy and a small phase lag in the lower frequencies. In a textbook by one of the authors, Hughes [4], the following list of properties that methods in structural dynamics should possess is presented.

1. Unconditional stability when applied to linear problems.
2. No more than one set of implicit equations to be solved at each step.
3. Second order accuracy.
4. Controllable algorithmic dissipation in the higher modes.
5. Self-starting.

It should be noted that unconditional stability in this case is related to the behaviour of the method when applied to the scalar test equation $y'' = -\omega^2 y$ (see [3]). The connection between conditions 1 and 3 is commented in [4]. Condition 2 suggests that multistep methods are preferable to for instance Runge-Kutta methods, and the theorem known as Dahlquist's second barrier then asserts that the maximal order is two under condition one. One can argue that condition 2 should be reconsidered. Frequently, one uses LU-factorization followed by a substitution algorithm to solve the implicit set of equations that appear in each step. It is well-known that the factorization requires most of the computational effort. Hence, if a small number of substitutions are performed in each step, the additional computational effort involved in the linear algebra part of taking one step is not substantial.

A number of authors discuss a variant of the Newmark family of methods

$$\begin{aligned}
 Ma_{n+1} + C((1 + \alpha)v_{n+1} - \alpha v_n) + K((1 + \alpha)d_{n+1} - \alpha d_n) &= F(t_{n+1} + \alpha h) \\
 d_{n+1} &= d_n + hv_n + h^2\left(\left(\frac{1}{2} - \beta\right)a_n + \beta a_{n+1}\right) \\
 v_{n+1} &= v_n + h((1 - \gamma)a_n + \gamma a_{n+1})
 \end{aligned} \tag{4}$$

where d_n and v_n are approximations to $y(t_n)$ and $y'(t_n)$, respectively (see [3]). The Newmark family is obtained with $\alpha = 0$. The only second order member of this family with $\alpha = 0$ is the familiar trapezoidal rule which is obtained with $\gamma = 1/2$ and $\beta = 1/4$. Despite its order of accuracy, the trapezoidal rule lacks the desired dissipation properties for higher frequencies. In [4] it is pointed out that the above method has order two if $\gamma = 1/2 - \alpha$ and $\beta = (1 - \alpha)^2/4$. It is shown in [3] that α can be used to control the damping of higher modes in the solution. Unconditional stability is ensured if $-\frac{1}{3} \leq \alpha \leq 0$ which, according to [3] is the range of practical interest. The variation of α provides a control device for algorithmic dissipation in the higher modes. Thus, one can say that the conditions 1-4 can be met. As for condition 5, one must address the fact that the method (4) is a multistep method when $\alpha \neq 0$. A starting method is therefore needed. The authors of [3] provide such a device through the formula

$$a_0 = M^{-1}(F(t_0) - Kd_0 - Cv_0).$$

This formula used in conjunction with (4) has order of consistency one if $\alpha \notin \{-\frac{1}{2}, 0\}$, as can be seen by comparing the Taylor expansions for the resulting one-step method to that of the exact solution. For instance, consider the special case where (2) is one-dimensional and $F(t) \equiv 0$. Comparing the two Taylor series yields

$$|d_1 - d(h)| = \mathcal{O}(h^3)$$

$$|v_1 - v(h)| = -\frac{1}{2}\alpha(1 + 2\alpha)\frac{CKd_0 + (C^2 - MK)v_0}{M^2}h^2 + \mathcal{O}(h^3)$$

Thus in general, if several restarts are required, the global order will reduce to one. The method (4) can be identified as a constant stepsize two-step Runge-Kutta-Nyström method. Thus, for variable stepsize implementations, the order will in general not be maintained. It has been pointed out, for instance by Mathisen [7], that variable stepsize should be allowed for.

Next, we find that there are reasons to question the quality of the damping mechanism in (4). Components of high frequencies are damped by a factor of 0.6 ($\alpha = -0.3$) in each time step. If at some time, say t^* , a high mode is injected into the numerical solution, it will be seen over several time steps before it is damped. We believe that such undesired frequency components should be damped instantly.

To this end, we are suggesting methods which belong to the large class of Runge-Kutta methods. We aim at fulfilling the conditions cited above, except that we allow for more than one substitution in the linear algebra part. We thereby gain the property of variable stepsize and instant damping of higher modes. Moreover, as these methods are one-step methods, no auxiliary formula is needed for restarts. The properties of the methods presented will be compared to those of (4).

2 A family of Runge-Kutta methods

2.1 SDIRK methods

The family of Runge-Kutta methods we consider here belongs to the celebrated class of SDIRK methods, which are discussed in detail in for instance [6]. Usually, such methods are formulated for initial value problems of the form $w' = f(t, w)$, $w(t_0) = w_0$ where w and w_0 are m -vectors, and $f : R \times R^m \rightarrow R^m$ is generally non-linear. We shall consider methods of the form

$$\left. \begin{aligned} k_r &= f(t_n + c_r h, w_n + h \sum_{j=1}^r a_{rj} k_j), \quad r = 1, \dots, s \\ w_{n+1} &= w_n + h \sum_{r=1}^s b_r k_r \end{aligned} \right\} \quad (5)$$

where h is the step size. We shall call s the number of *stages* of the method. In SDIRK methods $a_{11} = \dots = a_{ss} := \gamma \neq 0$, and we impose the additional condition that $\sum_{j=1}^r a_{rj} = c_r$ which implies $c_1 = \gamma$.

A *Butcher tableau* can be used to represent these methods. It has the form

$$\begin{array}{c|ccc} \gamma & \gamma & & \\ c_2 & a_{21} & \gamma & \\ \vdots & \vdots & & \ddots \\ c_s & a_{s1} & a_{s2} & \cdots & \gamma \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

We denote by A the lower triangular $s \times s$ matrix whose non-zero elements are $((a_{rj}))$, $1 \leq j \leq r \leq s$ and by b the s -vector (b_r) . We shall later restrict the attention to SDIRK methods being *stiffly accurate*, which means that $a_{sj} = b_j$, $j = 1, \dots, s$.

The name SDIRK is an acronym for *singly-diagonal-implicit Runge-Kutta* methods. The popularity of these methods can be explained by their ability to combine good stability properties with a relatively cheap implementation. If we apply an SDIRK method to (2), we obtain after some algebraic manipulation the following formulas to be used in each step: First form the matrix

$$T := M + h\gamma C + (h\gamma)^2 K$$

Then, for $r = 1, \dots, s$ solve the systems

$$T \cdot \bar{k}_r = F(t_n + c_r h) - K \left(y_n + c_r h y'_n + h^2 \sum_{j=1}^{r-1} \bar{a}_{rj} \bar{k}_j \right) - C \left(y'_n + h \sum_{j=1}^{r-1} a_{rj} \bar{k}_j \right)$$

with respect to \bar{k}_r . Here $\bar{a}_{r,j} = \sum_{k=1}^r a_{rk} a_{kj}$. Then compute the approximations y_{n+1} and y'_{n+1} to $y(t_{n+1})$ and $y'(t_{n+1})$ respectively from the formulas

$$y_{n+1} = y_n + h y'_n + h^2 \sum_{r=1}^s \bar{b}_r \bar{k}_r$$

$$y'_{n+1} = y'_n + h \sum_{r=1}^s b_r \bar{k}_r$$

where $\bar{b}_r = \sum_{k=r}^s a_{kr} b_k$. Observe that the s linear systems of equations that have to be solved in each step share the same coefficient matrix, hence, only one LU -factorization is necessary to solve for $\bar{k}_1, \dots, \bar{k}_s$. It should also be noted that the structure of the matrix T is inherited from the structure of the matrices M, C and K . If for instance all these matrices are symmetric, T will be symmetric.

In order to compare the SDIRK methods to (4) in the same way as in [3], it is necessary to consider the second order scalar test equation

$$u'' = -\omega^2 u \tag{6}$$

where ω is a real number. For the SDIRK methods, we obtain the recurrence formula

$$\begin{bmatrix} u_{n+1} \\ u'_{n+1} \end{bmatrix} = R(hM) \begin{bmatrix} u_n \\ u'_n \end{bmatrix} \quad \text{where} \quad M = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \quad \text{and} \quad R(z) = 1 + zb^T(I - zA)^{-1}e$$

$R(z)$ is a rational function defined for all (complex)¹ z except $z = 1/\gamma$ which is its only pole. It is known as the *stability function* of the method. e is the s -vector $(1, 1, \dots, 1)^T$ and I is the $s \times s$ identity matrix.

As in [3] we will carry out the damping and phase lag analysis by considering the eigenvalues of $R(hM)$. The eigenvalues of M are $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$, and it is well known that the corresponding eigenvalues of $R(hM)$ are $\mu_1 = R(iy)$ and $\mu_2 = R(-iy)$ where $y := \omega h$. We shall say that the Runge-Kutta method is *I-stable* if for all real y , the stability function satisfies $|R(iy)| \leq 1$. This is equivalent to the definition of unconditional stability as defined in [3]. For SDIRK methods we can write

$$R(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are the polynomials,

$$P(z) = \det(I - zA + zeb^T) \quad \text{and} \quad Q(z) = (1 - \gamma z)^s.$$

In fact, for SDIRK methods where the order of the methods p satisfies $p \geq s - 1$ it can be shown that ([6], p.103)

$$P(z) = (-1)^s \sum_{j=0}^s L_s^{(s-j)}\left(\frac{1}{\gamma}\right) (\gamma z)^j$$

where

$$L_s(x) = \sum_{j=0}^s (-1)^j \binom{s}{j} \frac{x^j}{j!}$$

is the s -degree Laguerre polynomial.

I-stability is equivalent to the condition $E(y) = Q(iy)Q(-iy) - P(iy)P(-iy) \geq 0$ for all real y . Furthermore, a Runge-Kutta method is *A-stable* if $|R(z)| \leq 1$ for all z such that $\text{Re } z \leq 0$. It can be shown ([6, p.43]) that an SDIRK method is *A-stable* if and only if it is *I-stable* and $\gamma > 0$. Finally, a method is *L-stable* if it is *A-stable*, and if $R(\infty) = 0$. Such a property ensures instant damping in the limit as $y \rightarrow \infty$. An SDIRK method with $\gamma > 0$ is *L-stable*, if it is *A-stable* and stiffly accurate ([6, p.45]).

Following [8] we write

$$R(iy) = \exp[(-a(y) + ib(y))y]$$

where $a(y)$ and $b(y)$ are real-valued functions. Notice that the solution $\exp(i\omega t)$ of (6) is reproduced exactly if $a(y) \equiv 0$ and $b(y) \equiv 1$. Hence the frequency distortion depends on the size of $|b(y) - 1|$. As in [8] we define the phase lag to be the modulus of the leading term in the expansion of $b(y) - 1$, and we define the *dispersion order* to be the degree of the leading term in the expansion. It is straightforward to verify that such an expansion will exist in a neighborhood of $y = 0$ for all consistent Runge-Kutta methods. In this neighborhood one can write

$$b(y) - 1 = -y^{-1} \left(\frac{i}{2} \log \frac{R(iy)}{R(-iy)} + y \right) \quad (7)$$

where the log function is cut along the negative real axis.

¹If $R(z) = P(z)/Q(z)$ where P and Q are polynomials, then $R(hM) = [Q(hM)]^{-1} \cdot P(hM)$, defined for all hM not having $1/\gamma$ as an eigenvalue.

2.2 A two-stage method

There exist only two SDIRK methods with $s = 2$ which are A-stable, stiffly accurate and have order two. These methods are given below where $\gamma = 1 \pm \frac{1}{2}\sqrt{2}$.

$$\begin{array}{c|cc} \gamma & & \gamma \\ 1 & 1 - \gamma & \gamma \\ \hline & 1 - \gamma & \gamma \end{array}$$

It follows from the above discussions that both these methods are L -stable. We shall consider only the method with $\gamma = 1 - \frac{1}{2}\sqrt{2}$ since we do not want to consider methods which rely upon an analytic extension of the solution of the initial value problem beyond the current step.

The stability function for this method is

$$R(z) = \frac{(1 - 2\gamma)z + 1}{(1 - \gamma z)^2}.$$

It is straightforward to verify that the dispersion order of this method is two, and the phase lag is $\frac{1}{2}\sqrt{2} - \frac{2}{3}$. This is less than the phase lag in e.g. the trapezoidal rule. Notice, however, that there are no parameters available in such a method for controlling the dissipation.

2.3 A family of three stage methods of order 2

Let us now consider three-stage methods of the form

$$\begin{array}{c|ccc} \gamma & & & \gamma \\ \sigma + \gamma & \sigma & & \gamma \\ 1 & b_1 & b_2 & \gamma \\ \hline & b_1 & b_2 & \gamma \end{array}$$

These methods have order two if

$$b_1 = 1 - \gamma - \frac{\gamma^2 - 2\gamma + \frac{1}{2}}{\sigma} \quad \text{and} \quad b_2 = \frac{\gamma^2 - 2\gamma + \frac{1}{2}}{\sigma},$$

hence there is a two-parameter family of second order methods. The stability function is

$$R(z) = \frac{(3\gamma^2 - 3\gamma + \frac{1}{2})z^2 - (3\gamma - 1)z + 1}{(1 - \gamma z)^3}.$$

These methods are L -stable for all γ in the interval between the (only) two real roots of the polynomial $24\gamma^4 - 72\gamma^3 + 48\gamma^2 - 12\gamma + 1$, i.e. $\gamma \in [\alpha, \beta]$ where $\alpha \approx 0.1804$ and $\beta \approx 2.1856$

(see [6] for details). Thus, the damping profile can be controlled by varying γ within this interval (see Figure 1 curves 4 and 5).

Since the stability function $R(z)$ only depends on the parameter γ we conclude that the other parameter σ has no effect on linear homogeneous systems such as (2) with $F(t) = 0$. However, we can choose σ in such a way that the method has order three for nonlinear problems if it has order three for linear problems. This implies for instance, that the method yields order three for quadrature problems using any value of γ . The formula for σ is then

$$\sigma = -\frac{\gamma^3 - 3\gamma^2 + 2\gamma - \frac{1}{3}}{\gamma^2 - 2\gamma + \frac{1}{2}}$$

It is not difficult to show that the above formula has order three for non-linear problems if we also require that γ is a root of the polynomial

$$p(\gamma) = \frac{1}{6} - \frac{3}{2}\gamma + 3\gamma^2 - \gamma^3$$

$p(\gamma)$ has three real roots which are approximately 0.1590, 0.4359, 2.4051. Notice that only the middle root corresponds to an L -stable method.

The phase lag expansion is given as

$$b(y) - 1 = p(\gamma)y^2 + \mathcal{O}(y^4)$$

hence, if γ is not a root of $p(\gamma)$, then the dispersion order is two, and the phase lag is $|p(\gamma)|$. By choosing γ as the middle root above ($\gamma \approx 0.4359$), one obtains dispersion order 4 and a phase lag of approximately 0.01540.

2.4 A family of four stage methods of order 3

Our final example is a family of four-stage methods of the form

$$\begin{array}{c|ccc} \gamma & \gamma & & \\ \sigma + \gamma & \sigma & \gamma & \\ \mu + \nu + \gamma & \mu & \nu & \gamma \\ 1 & b_1 & b_2 & b_3 \quad \gamma \\ \hline & b_1 & b_2 & b_3 \quad \gamma \end{array}$$

To simplify the expressions we define $\phi = \mu + \nu$. These methods have order three if

$$\begin{aligned} b_1 &= \frac{(1 - \gamma)\sigma\phi - \sigma(1/2 - 2\gamma + \gamma^2) + 1/3 - 2\gamma + 3\gamma^2 - \gamma^3 - (1/2 - 2\gamma + \gamma^2)\phi}{\sigma\phi} \\ b_2 &= \frac{1/3 - 2\gamma + 3\gamma^2 - \gamma^3 - (1/2 - 2\gamma + \gamma^2)\phi}{\sigma(\sigma - \phi)} \\ b_3 &= -\frac{1/3 - 2\gamma + 3\gamma^2 - \gamma^3 - \sigma(1/2 - 2\gamma + \gamma^2)}{\phi(\sigma - \phi)} \end{aligned}$$

$$\nu = \frac{\left(1/6 - \frac{3}{2}\gamma + 3\gamma^2 - \gamma^3\right) \phi (\sigma - \phi)}{\sigma (\gamma^3 + (\sigma - 3)\gamma^2 + (2 - 2\sigma)\gamma - 1/3 + \sigma/2)}$$

and defines a three-parameter family of order three methods. The stability function is given by

$$R(z) = -\frac{(24\gamma^3 - 36\gamma^2 + 12\gamma - 1)z^3 + (-36\gamma^2 + 24\gamma - 3)z^2 + (24\gamma - 6)z - 6}{6(\gamma z - 1)^4}.$$

We have L-stability for $\gamma \in [\alpha, \beta]$ where $\alpha \approx 0.2236$ and $\beta \approx 0.5728$ (see [6] p. 106).

Unfortunately, it is not possible to obtain order four with only four stages (see [1]). However, we can choose σ and ϕ such that they satisfy two of the conditions for the method to have order four. If this is done, we obtain

$$\sigma = \frac{1/12 - \gamma + \frac{7}{2}\gamma^2 - 4\gamma^3 + \gamma^4}{1/6 - \frac{3}{2}\gamma + 3\gamma^2 - \gamma^3}$$

$$\phi = \frac{1/8 - \frac{4}{3}\gamma + 4\gamma^2 - 4\gamma^3 + \gamma^4}{1/6 - \frac{3}{2}\gamma + 3\gamma^2 - \gamma^3}$$

The phase lag expansion is

$$b(y) - 1 = q(\gamma)y^4 + O(y^6)$$

where $q(\gamma)$ is the polynomial

$$q(\gamma) = -4\gamma^5 + 16\gamma^4 - 14\gamma^3 + \frac{14}{3}\gamma^2 - \frac{2}{3}\gamma + \frac{1}{30}$$

This polynomial has three real roots, for γ approximately 0.1117, 0.5257 and 2.9371. Note that only the middle root gives an L-stable method. If γ is a root of $q(\gamma)$, the dispersion order will be 6, otherwise it will be four and the phase lag will be $|q(\gamma)|$.

It is also possible to equip this family of methods with an embedded formula of order 2 for error estimation and stepsize control (see [2]). Since this formula is only going to be used for error estimation it does not have to be L-stable. A simple example of such a method is

$$\begin{array}{c|cc} \gamma & & \gamma \\ \tau + \gamma & \tau & \gamma \\ \hline & \hat{b}_1 & \hat{b}_2 \end{array}$$

where

$$\hat{b}_1 = \frac{2\tau - (1 - 2\gamma)}{2\tau}$$

$$\hat{b}_2 = \frac{1 - 2\gamma}{2\tau}$$

Now γ and τ should be chosen according to the third order methods above such that no extra nonlinear systems need to be solved.

3 Comparison with the Newmark family of methods

In order to compare the SDIRK methods presented above with the Newmark family of methods we shall look at some of the numerical characteristics of the methods. As in [3] and [4] we plot the spectral radii of the methods. The spectral radius can be seen as a measure of stability and dissipation in the numerical method. For SDIRK methods the *spectral radius* is given by

$$\rho = |R(iy)|$$

where $y = h/T$, h is the stepsize and T the period. In Figure 1 we show the spectral radii of some members of the SDIRK and the Newmark α -family of methods. The curves are as follows:

1. Newmark with $\alpha = -0.1$ damping.
2. Newmark with $\alpha = -0.3$ damping.
3. SDIRK with two stages (see Section 2.2).
4. SDIRK with three stages and $\gamma = 0.19$ (see Section 2.3).
5. SDIRK with three stages and $\gamma = 1.0$.
6. SDIRK with four stages and $\gamma = 0.23$ (see Section 2.4).
7. SDIRK with four stages and $\gamma = 0.5728$.

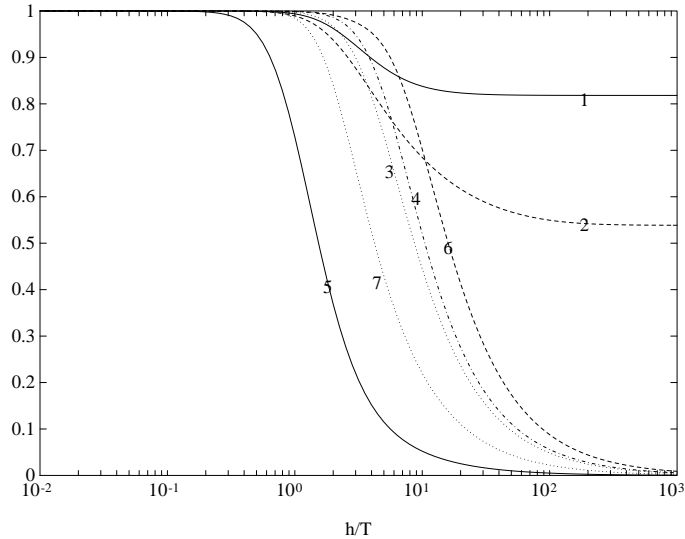


Figure 1: Spectral radii for SDIRK and Newmark methods with α damping.

For the SDIRK three and four stage methods we have chosen values for γ such that the method is L-stable. In curve 4 the value of γ is close to the lower value of its interval of L-stability. For curve 5 γ is chosen as 1 even though the upper value for an L-stable method is approximately 2.1856. This is because we do not want to consider methods

which rely on an analytic extension of the solution of the initial value problem beyond the current step. In curve 6 γ is chosen to be close to the left endpoint whereas in curve 7 γ is near the right endpoint of the interval of L-stability.

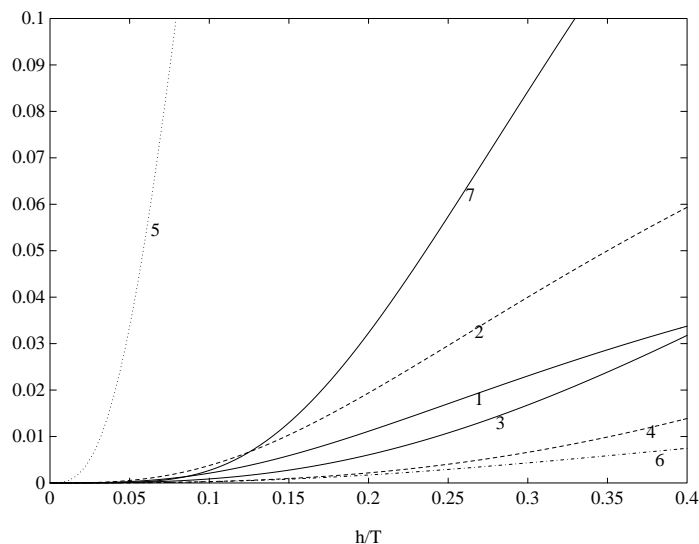


Figure 2: Algorithmic damping ratios for SDIRK and Newmark methods with α damping.

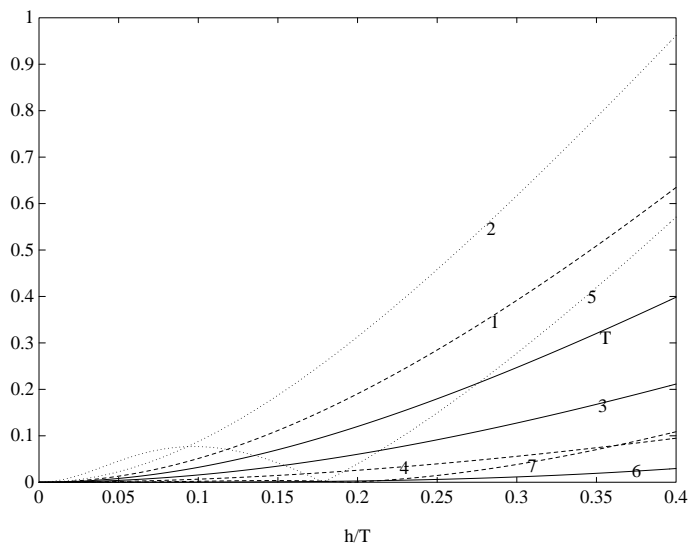


Figure 3: Relative period error for SDIRK and Newmark methods with α damping.

Note that both the three stage and the four stage family of methods show dissipation that can be controlled by the user by varying the γ parameter. For the three stage method the dissipation can be continuously varied between the two curves 4 and 5 by changing γ . In the same way the dissipation in the four stage method can be varied continuously between curves 6 and 7. Also note that the spectral radii of these methods eventually drops to zero since the methods are all L-stable.

In Figure 2 we plot the *algorithmic damping ratios* (see [3], [4]) for the same methods. For SDIRK methods this can be expressed as

$$\bar{\xi} = -\frac{\ln |R(iy)|}{\arctan\left(\frac{\text{Im}R(iy)}{\text{Re}R(iy)}\right)}.$$

Desirable properties for the algorithmic damping ratios are a zero slope at the origin and a controlled growth as h/T increases. All the graphs have a zero slope at the origin. In curves 5 and 7 most of the numerical dissipation is found at the lower modes. However, it should again be emphasized that the algorithmic damping ratios for the three stage and four stage family of methods can be controlled by varying the γ parameter.

In Figure 3 we present the *relative period errors* or $|(\bar{T} - T)/T|$ for the methods (see [3], [4]). The relative period error indicates how well the numerical method will follow the oscillations of the solution. For the SDIRK methods presented here this can be expressed as in (7).

Note that all the SDIRK methods except the three stage method with $\gamma = 1.0$ (curve 5) have smaller relative period errors than the Newmark α -methods. The relative period errors in the trapezoid method which is the best of the Newmark methods in this respect, is shown in the curve marked T. Curve 5 has a corner point between 0.17 and 0.18. This is caused by the change of sign of the relative period error of this SDIRK method. Recall that we plot the absolute value of the relative period error.

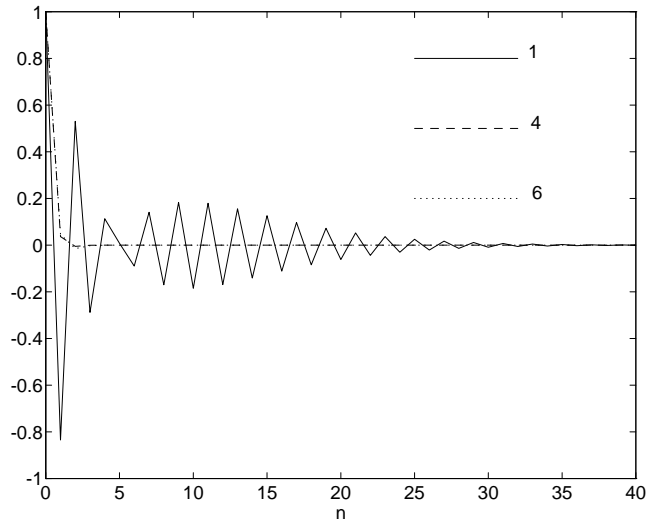


Figure 4: Comparison of overshoot response for curves 1, 4 and 6 (displacement).

Finally, we compare the “overshoot” properties (see [4, p. 537]) of some of the SDIRK methods with a member of the Newmark α -family. We have applied the methods to (6) with $\omega = 1$ and stepsize $h = 20\pi$, corresponding to a ratio $h/T = 10$. Figure 4 shows the displacement versus the step index for methods 1, 4 and 6 above while Figure 5 contains

the corresponding velocity curves. Curves 4 and 6 nearly coincide for both displacement and velocity. This behavior is typical for all SDIRK methods presented in this paper.

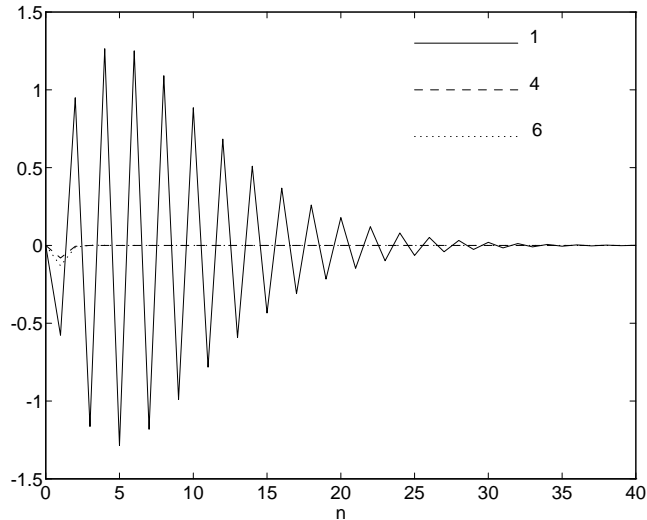


Figure 5: Comparison of overshoot response for curves 1, 4 and 6 (velocity).

4 Conclusion

Runge-Kutta methods suitable for time integration in structural dynamics have been presented. The methods belong to a subclass called SDIRK (singly-diagonal-implicit Runge-Kutta) methods. All the methods presented are stiffly accurate and L-stable. These methods which can be implemented with approximately the same cost as the Newmark methods with α -damping, have superior stability properties. The four stage method presented has third order accuracy whereas the Newmark methods have order two. The three and four stage methods presented have controllable algorithmic dissipation in the higher modes. All these methods have relative period errors that are smaller than or comparable to the Newmark methods. Error control is possible using embedded methods. One such method is constructed for the third order methods with four stages. All the SDIRK methods presented are self-starting due to their one-step nature.

It is the authors belief that SDIRK methods of the type presented here have properties that make them interesting candidates for numerical integration in structural dynamics. The four stage, third order method appears to be particularly suitable due to its higher order and superior dissipation and period error characteristics.

Since the Newmark method with α damping can be identified as a two-step Runge-Kutta-Nyström method, it would be interesting to expand the search to the class of two step Runge-Kutta methods. Hopefully, it will be possible to construct methods of this type that allow for a variable step size while retaining the desirable properties of for instance the Newmark method with α damping.

References

- [1] R. Alexander, *Diagonally Implicit Runge-Kutta methods for STIFF O.D.E.'s* SIAM J. Numer. Anal., Vol. 14, No. 6, pp. 1006–1021, 1977.
- [2] J. R. Cash, *Diagonally Implicit Runge-Kutta Formulae with Error Estimates*, J. Inst. Maths Applics., Vol. 24, pp. 293–301, 1979.
- [3] H. M. Hilber, T. J. R. Hughes, R. L. Taylor, *Improved numerical dissipation for time integration algorithms in structural dynamics*, Earthquake Engin. and Struct. Dynamics, Vol. 5, pp. 283–292, 1977.
- [4] T. J. R. Hughes, *The Finite Element Method, Linear Static and Dynamic Finite Element Analysis*, Prentice-Hall, 1987.
- [5] E. Hairer, S.P. Nørsett, G. Wanner, *Solving Ordinary Differential Equation I, Nonstiff Problems*, Springer, 1987.
- [6] E. Hairer, G. Wanner, *Solving Ordinary Differential Equations II, Stiff and Differential-Algebraic Problems*, Springer 1991.
- [7] K. M. Mathisen, *Large displacement analysis of flexible and rigid systems considering displacement-dependant loads and nonlinear constraints*, PhD thesis, The Norwegian Institute of Technology, 1990.
- [8] R. M. Thomas, *Phase properties of high order, almost P-stable formulae*, BIT 24, pp. 225–238, 1984.

SINTEF, INDUSTRIAL MATHEMATICS, N-7034 TRONDHEIM, NORWAY

E-mail address: Brynjulf.Owren@sima.sintef.no
H.H.Simonsen@usit.uio.no