

A uniqueness result related to the stability of explicit Runge-Kutta methods

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Abstract

The polynomial associated with the largest disk of stability of an m -stage explicit Runge-Kutta method of order p is unique.

This note should be considered as an addendum to [2].

Consider the set $\mathcal{P}_{m,p}$ consisting of polynomials of the form

$$P(\zeta) = \sum_{n=0}^m \alpha_n \frac{\zeta^n}{n!}$$

where $\alpha_0 = \cdots = \alpha_p = 1$ and $1 \leq p \leq m$. It is well known that when an m -stage explicit Runge-Kutta method of order p is applied to the test equation $y' = \lambda y$, $\lambda \in \mathbf{C}$ one obtains a stability polynomial of the above form. The stability region of such a polynomial is

$$S_P = \{\zeta \in \mathbf{C} : |P(\zeta)| \leq 1\}$$

Introducing the disk $D_r = \{\zeta \in \mathbf{C} : |\zeta + r| \leq r\}$ we define the functional $R_{m,p} : \mathcal{P}_{m,p} \rightarrow \mathbf{R}$ by

$$R_{m,p}(P) = \sup\{r : D_r \subset S_P\}.$$

By a compactness argument a maximum of $R_{m,p}(P)$ must exist, so we may define

$$\rho = \rho(m, p) = \max_{P \in \mathcal{P}_{m,p}} R_{m,p}(P).$$

Denote such an optimal polynomial (for which $R_{m,p}(P) = \rho$) by $\hat{P}_{m,p}$. We prove

Theorem 1 $\hat{P}_{m,p}$ is unique for any $m \geq p \geq 1$.

This theorem, which was conjectured in [3], is a simple consequence of the following lemma.

Lemma 2 *The stability region of any $\hat{P}_{m,p}$ touches the circle $\Gamma_\rho = \{\zeta : |\zeta + \rho(m,p)| = \rho(m,p)\}$ at least at $m - p + 2$ distinct points.*

Proof: It is sufficient to consider the case $p > 1$ since by [1], the lemma is true for $p = 1$. Assume that the stability region of $\hat{P}_{m,p}$ touches Γ_ρ only at $k \leq m - p + 1$ distinct points, $0 = \zeta_1, \zeta_2, \dots, \zeta_k$. We prove that this contradicts $\hat{P}_{m,p}$ being optimal. Introduce the polynomial

$$Q(\zeta) = \zeta^{m-k+2} \sum_{j=2}^k a_j \prod_{\substack{l=2 \\ l \neq j}}^k \frac{\zeta - \zeta_l}{\zeta_j - \zeta_l}$$

where

$$a_j = \frac{\hat{P}_{m,p}(\zeta_j)}{\zeta_j^{m-k+2}}$$

Since $Q(\zeta_j) = \hat{P}_{m,p}(\zeta_j)$, $j = 2, \dots, k$ we have for small $\epsilon > 0$

$$|\hat{P}_{m,p}(\zeta) - \epsilon Q(\zeta)| < |\hat{P}_{m,p}(\zeta)|$$

in sufficiently small neighbourhoods of each ζ_2, \dots, ζ_k . On Γ_ρ close to ζ_1 we have

$$|P(z(\theta))|^2 = 1 - \rho\theta^2 + \mathcal{O}(\theta^3)$$

for any $P \in \mathcal{P}_{m,2}$ where we have put $z(\theta) = -\rho + \rho e^{i\theta} = \rho i\theta - \frac{1}{2}\rho\theta^2 + \mathcal{O}(\theta^3)$. Thus $|P(z)| < 1$ for small $z \in \Gamma_\rho \setminus \{0\}$. On the remaining part of Γ_ρ , say $\Gamma_\rho^{(0)}$ we compute $\delta > 0$ such that

$$\sup_{\zeta \in \Gamma_\rho^{(0)}} |\hat{P}_{m,p}(\zeta)| = 1 - \delta$$

Choosing

$$\epsilon = \frac{\delta}{2} \frac{1}{\sup_{\zeta \in \Gamma_\rho^{(0)}} |Q(\zeta)|}$$

we finally obtain

$$|\hat{P}_{m,p}(\zeta) - \epsilon Q(\zeta)| < 1$$

for all $\zeta \in \Gamma_\rho \setminus \{0\}$ which is a contradiction. \square

Proof of the theorem: Assume there are two distinct polynomials \hat{P} and \check{P} which are both optimal. Then so is clearly $\frac{1}{2}(\hat{P} + \check{P})$. Since $|\frac{1}{2}(\hat{P} + \check{P})| = 1$ at least at $m - p + 2$ distinct points on Γ_ρ , it follows that $\hat{P}(\zeta) - \check{P}(\zeta)$ has at least $m - p + 2$ zeros on Γ_ρ . But this is a contradiction since

$$\hat{P}(\zeta) - \check{P}(\zeta) = \zeta^{p+1}(c_0 + \cdots + c_{m-p-1}\zeta^{m-p-1})$$

has at most $m - p$ distinct zeroes. \square

Remark: Theorem 6 of [2] is now an immediate consequence of the above theorem in conjunction with the (sharp) upper bound given by Proposition 4 of [2].

References

- [1] R. Jeltsch and O. Nevanlinna, Largest disk of stability of explicit Runge-Kutta methods, BIT 18 (1978), pp 500-502.
- [2] B. Owren and K. Seip, Some stability results for explicit Runge-Kutta methods, BIT (to appear).
- [3] R. van der Marel, Stability Radius of Polynomials Occurring in the Numerical Solution of Initial Value Problems, Report TW-89-04, University of Leiden (1989).