Norwegian University of Science and Technology
Department of Mathematical Sciences

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Exam in TMA4225 Foundation of Analysis

English
Tuesday December 4, 2007
Time: 09.00 - 13:00
Permitted aids: Code B

Grades to be announced: Friday December 14, 2007

Problem 1

a) Give an example of a sequence of integrable functions $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}$ taking values ≥ 0, such that $f_n(x) \to 0$ for almost all x, but $\int f_n \nrightarrow 0$, as $n \to \infty$.

b) Is it possible to find a sequence $\{f_n\}_{n=1}^{\infty}$ satisfying the conditions in a), with the added requirement that $f_n(x) \leq e^{-x^2}$ for all x and all n? (Give reasons.)

Problem 2 Prove that if $f\colon \mathbf{R}^d \to \mathbf{R} \cup \{\pm\infty\}$ is integrable and $\displaystyle\int_E f(x)dx \geq 0$ for every measurable set E, then $f(x) \geq 0$ for a.e. x.

Problem 3

a) Let $f\colon [0,1] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ \frac{1}{q} \text{ if } x = \frac{p}{q} \in \mathbf{Q} \ (p \text{ and } q \text{ are relatively prime}). \end{cases}$$

Prove that f is discontinuous at x if and only if $x \in \mathbf{Q}$.

b) The function f in a) is Riemann integrable on $[0,1]$ (explain why). Compute the Riemann integral $\int_{[0,1]}^{R} f(x)dx$.

Problem 4

a) Let the absolutely continuous function $F\colon [0,1] \to \mathbf{R}$ be defined by

$$F(x) = \int_0^x \chi_K(t)dt$$

where $K = [0,1] - C$, and C is a generalized (or Cantor-like) subset of $[0,1]$ such that $m(C) > 0$. (χ_K is the characteristic function of K.) Show that $F(x)$ is strictly increasing, i.e. $x < y \Rightarrow F(x) < F(y)$.

b) Show that the derivative of F is zero on a measurable subset E of $[0,1]$, where $m(E) > 0$.

Problem 5

a) Let $h\colon A \to [0,\infty]$ be integrable, where $A \subset \mathbf{R}^d$ is measurable, and let $E = \{(x,y) \in A \times \mathbf{R} | 0 \le y \le h(x)\}$, i.e. E is the subgraph of h. Show that E is a measurable subset of $\mathbf{R}^{d+1}$ and that

$$\int_A h(x)dx = m(E)$$

where $m = m_{\mathbf{R}^{d+1}}$ is the Lebesgue measure on $\mathbf{R}^{d+1}$.

b) Let $g\colon [0,\pi] \to \mathbf{R}$ be defined by

$$g(x) = \int_x^\pi \frac{\sin t}{t} dt.$$

By considering the function $F\colon [0,\pi] \times [0,\pi] \to \mathbf{R}$ defined by $F(x,t) = \frac{\sin t}{t}\chi_{[x,\pi]}(t)$, and using Fubini's Theorem, show that

$$\int_0^\pi g(x)dx = 2.$$