TMA4225 FOUNDATIONS OF ANALYSIS, FALL 2008

Homework 8

Problem 1. Define f on $[0,1]$ by $f(0) = 0$ and $f(x) = x^a \sin(1/x)$ for $x \neq 0$. Show that f is absolutely continuous on $[0,1]$ if and only if $a > 1$.

Problem 2. A function f is said to satisfy a Lipschitz condition on $[a,b]$ if there is a constant M such that

$$|f(x) - f(y)| \leq M|x - y|, \quad x, y \in [a,b]\,.$$

a) Show that if f has a bounded derivative on $[a,b]$, then f satisfies a Lipschitz condition on $[a,b]$.
b) Prove that if f satisfies a Lipschitz condition on $[a,b]$, then it is absolutely continuous on $[a,b]$.
c) Show that the converse of b) does not hold by proving the following: The function $f(x) = \sqrt{x}$ is absolutely continuous on $[0,1]$, but does not satisfy a Lipschitz condition on $[0,1]$.

Problem 3. S&S p. 149: 20(a)