

LIMSUP, LIMINF AND ACCUMULATION POINTS

TMA4225 FOUNDATIONS OF ANALYSIS, FALL 2008

In this note the main discussion will be about \limsup etc. for sequences of real numbers. The corresponding concepts for functions of a real variable will be mentioned briefly at the end.

SEQUENCES OF REAL NUMBERS

We begin by recalling the definition of \limsup (limit superior) and \liminf (limit inferior) for a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ (cfr. Stein and Shakarchi, p. 29):

Definition 1.

$$\limsup_{n \rightarrow \infty} a_n = \inf_k (\sup_{n \geq k} a_n) \quad \liminf_{n \rightarrow \infty} a_n = \sup_k (\inf_{n \geq k} a_n).$$

\limsup and \liminf always exist as extended real numbers. Since the sequence $b_k = \sup_{n \geq k} a_n$ is decreasing, $\lim_k b_k$ exists (possibly as an infinite number), thus we may also write $\inf_k (\sup_{n \geq k} a_n) = \lim_k (\sup_{n \geq k} a_n)$ (hence the notation \limsup). A similar remark holds for \liminf .

For bounded sequences (see below), \limsup and \liminf always exist as (finite) real numbers.

We recall the definition of an *accumulation point*:

Definition 2. A point $c \in \mathbf{R}$ is said to be an *accumulation point* for $\{a_n\}_{n=1}^{\infty}$ if every neighborhood¹ of c contains infinitely many terms from the sequence $\{a_n\}_{n=1}^{\infty}$; i.e.: for every $\epsilon > 0$ and every natural number N there is an index $n > N$ such that $|c - a_n| < \epsilon$. We say that $+\infty$ (resp. $-\infty$) is an accumulation point if for every $M \in \mathbf{R}$ and every natural number N there is an index $n > N$ such that $a_n > M$ (resp. $a_n < M$).

Proposition 1. A point $c \in \mathbf{R}$ is an accumulation point for $\{a_n\}$ if and only there is a subsequence of $\{a_n\}$ which converges to c .

(A similar statement holds if $|c| = \infty$.)

Proof. If there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim_{k \rightarrow \infty} a_{n_k} = c$, then for every $\epsilon > 0$ there is a natural number K such that $|c - a_{n_k}| < \epsilon$ for all $k \geq K$. Since $\{n_k\}_{k \geq K}$ is an infinite set of indices, c is an accumulation point for $\{a_n\}$. Conversely, if c is an accumulation point, then for each $k \in \mathbf{N}$ we can find an index n_k such that $|c - a_{n_k}| < 1/k$. Since each n_k can be chosen arbitrarily large, we can arrange it so that the sequence of indices $\{n_k\}$ is strictly increasing. The subsequence $\{a_{n_k}\}$ then clearly converges to c . \square

Bounded sequences. Recall that a sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *bounded* if all the terms a_n stay within some finite interval; or equivalently: if there is an $R > 0$ such that $|a_n| \leq R$ for all n .

Theorem 1. A bounded sequence has at least one accumulation point.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence and assume $|a_n| \leq R$ for all n . The proof is done by repeatedly bisecting the interval $[-R, R]$. Set $I_0 = [-R, R]$ and split I_0 into two closed subintervals by bisecting it. At least one of the two subintervals must contain infinitely many terms of the sequence. If only one of them contains infinitely many terms, let I_1 be the subinterval with infinitely many terms; if both subintervals have infinitely many terms, let I_1 be the left subinterval, for example. Now bisect I_1 , and select I_2 according to the same rule. Continuing this process we obtain a nested sequence of closed intervals $\{I_n\}$ such that each I_n contains infinitely terms from the sequence $\{a_n\}$. Since all the finite intersections $\bigcap_{n=1}^N I_n = I_N$ are nonempty, and since the intervals I_n are closed, it follows from the compactness of $[-R, R]$ that the infinite intersection

¹By a neighborhood of a point we mean an open interval containing that point.

$\bigcap_{n=1}^{\infty} I_n$ is nonempty, too. Furthermore, since $|I_n| = R/2^n \rightarrow 0$ as $n \rightarrow \infty$, the infinite intersection $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one point; call it c . This c is an accumulation point, because for any $\epsilon > 0$ we have $I_n \subset (c - \epsilon, c + \epsilon)$ for $n > \log(R/\epsilon)/\log 2$, and I_n contains infinitely many terms from $\{a_n\}$ by construction. \square

Proposition 2. *A bounded sequence $\{a_n\}_{n=1}^{\infty}$ converges if and only if $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$, in which case $\lim_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$.*

Proof. Set $b_k = \sup_{n \geq k} a_n$ and $b'_k = \inf_{n \geq k} a_n$. Then $b'_k \leq a_k \leq b_k$ for all k .

If $\{a_n\}_{n=1}^{\infty}$ converges and $c = \lim_{n \rightarrow \infty} a_n$, then for every $\epsilon > 0$ there is a natural number K such that $c - \epsilon < a_k < c + \epsilon$ for all $k \geq K$. Since $a_k \leq \sup_{n \geq k} a_n = b_k$ and $a_k \geq \inf_{n \geq k} a_n = b'_k$ we have $c - \epsilon \leq b'_k \leq a_k \leq b_k \leq c + \epsilon$ for all $k \geq K$. It follows that $\lim_k b'_k = \lim b_k = c$, i.e., $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n$.

Conversely, if $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$, then since $b'_k \leq a_k \leq b_k$, we get $\lim_k a_k = \lim_k b_k = \lim_k b'_k$ by the squeeze law, i.e., $\lim_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$. \square

Theorem 2. *If $\{a_n\}$ is a bounded sequence, then its set of accumulation points is a compact subset of \mathbf{R} . Furthermore, $\limsup a_n$ (resp. $\liminf a_n$) is the largest (resp. the smallest) accumulation point of $\{a_n\}$.*

Proof. Let C denote the set of accumulation points for $\{a_n\}$. Then C is bounded (since $\{a_n\}$ is bounded), so to prove it is compact, it suffices to prove that it is closed. To that end, let $\{c_n\}$ be a sequence from C such that $\lim_n c_n = c \in \mathbf{R}$. We must prove that $c \in C$, i.e., that every neighborhood of c contains infinitely many terms from the sequence $\{a_n\}$. So let U be a neighborhood of c . There is an N such that $c_n \in U$ for $n \geq N$. Since U is a neighborhood of c_N , too, and since c_N is an accumulation point, it contains infinitely many terms from $\{a_n\}$. Thus $c \in C$, and it follows that C is closed.

For the second statement it suffices to prove $\limsup a_n$ is an accumulation point, and that $\limsup a_n \geq c$ for any $c \in C$, as the corresponding proofs for \liminf are completely analogous.

First we verify (the rather obvious fact) that \limsup is an accumulation point. Set $L = \limsup a_n$, let U be a neighborhood of L , and set $b_k = \sup_{n \geq k} a_n$. There is a K such that $b_k \in U$ for all $k \geq K$. Since $b_k = \sup_{n \geq k} a_n$, every neighborhood of b_k contains an a_{n_k} with $n_k \geq k$. Now U is a neighborhood of b_k for all $k \geq K$, thus it contains terms a_{n_k} with arbitrarily large index n_k .

Finally we prove that if c is any accumulation for $\{a_n\}$, then $\limsup a_n \geq c$. By Proposition 1 there is a subsequence $\{a_{n_k}\}$ which converges to c . This gives:

$$\begin{aligned} c &= \lim_{k \rightarrow \infty} a_{n_k} = \inf_m \left(\sup_{k \geq m} a_{n_k} \right) \\ &= \inf_m \left(\sup \{ a_{n_m}, a_{n_{m+1}}, a_{n_{m+2}}, a_{n_{m+3}}, \dots \} \right) \\ &\leq \inf_m \left(\sup \{ a_{n_m}, a_{n_m+1}, a_{n_m+2}, a_{n_m+3}, \dots \} \right) \\ &= \limsup a_n \end{aligned}$$

\square

Example. Set $a_n = (-1)^n$. Then $\{a_n\}$ has exactly two accumulation points, namely 1 and -1 . Further: $\limsup_n a_n = 1$, $\liminf_n a_n = -1$. The subsequence $a_{2k} = (-1)^{2k} = 1$ converges to $1 = \limsup_n a_n$, and the subsequence $a_{2k-1} = (-1)^{2k-1} = -1$ converges to $-1 = \liminf_n a_n$.

FUNCTIONS OF A REAL VARIABLE

Definition 3. Let f be a function of a real variable, defined in a neighborhood of a point x_0 . Then we define:

- Two-sided lim sup: $\limsup_{x \rightarrow x_0} f(x) = \inf_{\delta > 0} (\sup_{0 < |x - x_0| < \delta} f(x))$
- lim sup from the right: $\limsup_{x \rightarrow x_0^+} f(x) = \inf_{\delta > 0} (\sup_{0 < x - x_0 < \delta} f(x))$
- lim sup from the left: $\limsup_{x \rightarrow x_0^-} f(x) = \inf_{\delta > 0} (\sup_{0 < x_0 - x < \delta} f(x))$

with similar definitions for \liminf .

Proposition 3. $\lim_{x \rightarrow x_0} f(x)$ exists if and only if $\limsup_{x \rightarrow x_0} f(x) = \liminf_{x \rightarrow x_0} f(x)$, in which case all three quantities are equal. Similar statements hold for the one-sided limits.

Proof. Analogous to the proof of Proposition 2. □